IL
NUOVO CIMENTO
ORGANO DELLA SOCIETÀ ITALIANA DI FISICA
SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

VOL. IX, N. 6 *Serie decima* 16 Settembre 1958

On a Possible Limit of Applicability of Quantum Electrodynamics.

D. I. BLOHINČEV

Joint Institute for Nuclear Research, Laboratory for Theoretical Physics - Moscow

(ricevuto il 17 Marzo 1958)

Summary. — In the present note the processes competing with the electromagnetic ones at high energies are considered. It is shown that such processes may be those involving the four-fermion interactions.

1. – Introduction.

It has been shown (1) that the application of the modern renormalization method in quantum electrodynamics leads to a difficulty of principle, *i.e.*, to the vanishing of the renormalized charge. Although the absolute proof of this conclusion was argued (2), none the less, the presence of the principle difficulties in the energy region E determined by the condition $\alpha \ln(E/mc^2) \sim 1$ ($\alpha = e^2/\hbar c$) appears to be rather convincing. The space scale $l_0 \sim (\hbar/mc) \cdot \exp[-3\pi/\alpha]$ corresponding to this energy is far beyond the limits of the gravitational radius of the electron as it was first pointed out in (3). The extreme energy itself is enormously high ($E_0 \sim mc^2 \cdot \exp[3\pi/\alpha]$).

(1) L. D. LANDAU, A. A. ABRIKOSOV and I. M. HALATNIKOV: *Dokl. Akad. Nauk SSSR*, **95**, 1177 (1954).

(2) N. N. BOGOLJUBOV and D. V. ŠIRKOV: *Introduction to Quantum Field Theory* (Moscow, 1957), pp. 355-356.

(3) M. A. MARKOV: *Žu. Éxper. Teor. Fiz.*, **17**, 661 (1947).

It can be expected, therefore, that the limits of applicability of modern electrodynamics will be revealed much earlier, *e.g.*, due to a possible change of space-time structure in space-time regions which are small but still considerably greater than l_0.

There is, however, another possibility of limiting the significance of quantum electrodynamics which is more accessible to theoretical analysis.

Together with the purely electrodynamic interactions of photons, electrons and positrons there occur the processes involving mesons and nucleons. They may be induced in a purely electrodynamic way, *e.g.*, by photon interaction with an electron.

If it would turn out that the contribution of these non-electromagnetic processes exceed that of the electromagnetic ones then it would not be possible to consider pure electrodynamics without essentially involving other types of interactions. In particular, from a certain energy E_{KT} the expansion in a series $e^2/\hbar c$ would become not reasonable.

We will show that such a competing interaction may be the weak four-fermion Fermi interaction.

The validity of this interaction in the high energy region is not experimentally checked and different theoretical doubts on the applicability of this interaction for the energies $E \gg mc^2$ may arise.

However, we shall start from the assumption about the applicability of this interaction up to very high energies and consider the conclusions resulting from this assumption.

The physical property of the pure fermion interactions is that their matrix elements do not decrease with the increase of the energy of the fermions involved in the process, while the matrix elements of the processes involving bosons (photons, π and K-mesons) decrease while the boson energy increases. It can be accounted for the fact that the boson field falls as $K^{-\frac{1}{2}}$ with increasing boson energy as follows:

$$\Phi_K = \sqrt{\frac{\hbar}{2K}} \exp\,[iKx] b_K^+ + \text{conj.}$$

where K is the boson momentum, b_K^+ is the operator of the boson production; while the fermion field is constant with increasing fermion energy:

$$\psi_K \sim u_K \exp\,[iKx] a_K^+ + \text{conj.},$$

where u_K is the spinor amplitude, a_K is the operator of fermion production.

We shall show further that due to this property the fermion interactions become essential in the electromagnetic processes much earlier than the energy reaches the logarithmic limit $E \sim mc^2 \exp\,[3\pi/\alpha]$.

2. – Fermion-electromagnetic interaction.

Let us consider the interaction process of a photon (K) with an electron (e) leading to the production of a μ-meson (μ) and two neutrinos (ν, $\tilde{\nu}$):

$$\mathrm{K} - \mathrm{e} \to \mu + \nu + \tilde{\nu} \,. \tag{1}$$

Such a process will be described by the interaction Lagrangian

$$W = eW_e + eW_\mu + gW_{e\mu\nu} \,, \tag{2}$$

where $eW_e = (I_e A)$ is the electron interaction (I_e is the electron current) with the electromagnetic field (A is the vector potential), eW_μ has the same meaning for a meson. Finally, $gW_{e\mu\nu}$ is the four-fermion interaction of an electron, a μ-meson and a neutrino:

$$g = \hbar c \Lambda_0^2 \simeq 10^{-43} \,\mathrm{erg \cdot cm^3}$$

is the Fermi constant, $\Lambda_0 = 6 \cdot 10^{-17}$ cm^4, whereas

$$W_{e\mu\nu} = (\overline{\psi}_\mu O_1 \psi_e)(\overline{\psi}_\nu O_2 \psi_\nu) + \mathrm{conj.}$$

Here ψ_e, ψ_μ, ψ_ν are the spinor electron, meson and neutrino fields, respectively; O_1 and O_2 are certain spinor operators.

The total effective cross-section for the process (1) is:

$$\sigma_\mu = \frac{2\pi}{\hbar c} \int |W_{af}|^2 \frac{P_\nu^2 \,\mathrm{d}P_\nu \,\mathrm{d}\Omega_\nu \tilde{P}_\nu'^2 \,\mathrm{d}\tilde{P}_\nu' \,\mathrm{d}\tilde{\Omega}_\nu'}{(2\pi\hbar)^6 \,\mathrm{d}E_f} \,, \tag{3}$$

where W_{af} is the matrix element from the interaction energy (2) for process (1) P_ν, $\tilde{P}_\nu$ are neutrino and antineutrino momenta, E_f is energy of the finite state.

The structure of this matrix element W_{af} is such that in the first non-vanishing approximation it is equal to

$$W_{af} = eg \sum_j \left\{ \frac{(a|We|c)(c|We\mu\nu|f)}{E_0 - E_c} + \frac{(a|We\mu\nu|c)(c|W\mu|f)}{E_0 - E_c'} \right\} , \tag{4}$$

where E_0 is the energy of the initial state, and E_c is the intermediate state energy. In the system of the center of gravity of a photon and electron $E_0 - E_c \sim \hbar c K$ (K is the photon wave vector),

$$(a|W_e|c) \sim K^{-\frac{1}{2}} \,, \qquad (c|W_\mu|f) \sim K^{\frac{1}{2}} \,.$$

Therefore, $|W_{af}|^2 \sim egK^{-3}$. The weight factor in (3) is proportional to K^5 Thus, the total cross-section is as follows

$$\sigma_{\mu} \cong \alpha \Lambda_{\nu}^4 K^{\cdot} \cdot F \,, \tag{5}$$

where F is a factor of order 1, which weakly depends on K (*).

Just in a similar manner one may consider the collision of two electrons and their simultaneous conversion into two mesons, in accordance with the scheme

$$\mathrm{e}' + \mathrm{e}'' \to \mu' + \mu'' \,. \tag{1'}$$

The differential cross-section (in the center of mass system) for this process will be

$$\mathrm{d}\sigma_{\mu\mu} \cong \Lambda_{\nu}^8 q^4 P^2 F \cdot \mathrm{d}\Omega \,, \tag{6}$$

where q is the momentum transfer and P is the initial electron momentum, both measured in reciprocal lengths.

On the other hand, the cross-sections of purely electromagnetic processes are equal to

$$\sigma = \frac{1}{2} \pi \alpha^2 \frac{1}{K^2} \left(\ln \frac{4K^2}{K_c^2} + \frac{1}{2} \right) \tag{7}$$

for the Compton effect,

$$\mathrm{d}\sigma_{\mathrm{ee}} = \alpha^2 \frac{P^2}{q^4} \, \mathrm{d}\Omega \tag{8}$$

for electron elastic collision,

$$\sigma_{\mathrm{p}} = \frac{28}{9} \alpha^3 \frac{1}{K^2} \left(\ln \frac{4K^2}{K_c^2} - 3.5 \right) \tag{9}$$

for pair production (here $K_c = (mc/\hbar)$, $\alpha = (e^2/\hbar c)$) and

$$\sigma_{\gamma} = 4\alpha^3 \frac{1}{K_c^2} \left| \ln \frac{4}{K_c^2} - 3.5 \right| \tag{10}$$

for the bremsstrahlung in the electron collision.

(*) These qualitative conclusions are supported by more detailed calculations made by Dr. M. Mayer (Rumania). The author is very grateful to him.

The comparison of these cross-sections with those of mixed processes (1) and (1′) shows that

$$\sigma_{\mu} > \sigma_{e} \qquad \text{with} \qquad K \gtrsim \alpha^{\frac{1}{2}} \frac{1}{\Lambda_0}, \tag{11}$$

$$\sigma_{\mu} > \sigma_{p} \qquad \text{with} \qquad K \gtrsim \alpha^{\frac{1}{2}} \left(\frac{\alpha}{\Lambda_0 K_c}\right) \frac{1}{\Lambda_0}, \tag{12}$$

$$\mathrm{d}\sigma_{\mu\mu} > \mathrm{d}\sigma_{ee} \qquad \text{with} \qquad q \gtrsim \alpha^{\frac{1}{2}} \frac{1}{\Lambda_0}, \tag{13}$$

$$\sigma_{\mu\mu} > \sigma_{\gamma} \qquad \text{with} \qquad q \sim p \geqslant \left(\frac{\alpha^3}{K^2 \Lambda_0^2}\right) \frac{1}{\Lambda_0}. \tag{14}$$

Here the factors ~ 1 are omitted (*).

We see from these inequalities that if four-fermion interactions may be considered applicable in the energy range $K > 1/\Lambda_0$, the processes with neutrinos and μ-meson production are more intensive than the purely electromagnetic processes. The corresponding photon and electron energy in the system of center of gravity should be greater than $(\hbar c/\Lambda_0) \sim 2$ GeV.

This is a great energy but, nevertheless, it is much lower than the logarithmic one.

It should be noted that the production of nucleon and meson pairs will play a considerably smaller role, since their production cross-section will be $(m/M)^2$ times less than that of electron-positron pairs.

The processes involving the production of neutrinos and boson mesons will be essential later, due to the above-mentioned difference in the behaviour of the boson and fermion matrix elements.

Thus, the fermion interaction may be the one which restricts the region of applicability of electrodynamics by a scale $> \Lambda_0$. For smaller scales and, consequently, for energies above $\hbar c/\Lambda_0$ it is not reasonable at all to study electrodynamics without considering the processes involving both μ-mesons and neutrinos and the Fermi constant g, together with $e^2/\hbar c$.

(*) Note that pair production in this energy range is the main electromagnetic process.

RIASSUNTO (*)

Si considerano nella presente nota i processi che alle alte energie competono con quello elettromagnetico. Si dimostra che tali processi possono essere quelli che interessano le interazioni fra quadrifermioni.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

On the Theory of the Linear Betatron.

M. V. Konjukov

The Tula Pedagogical Institute - Tula

J. P. Terletskij

The Physical Faculty of the Moscow State University - Moscow

(ricevuto il 25 Marzo 1958)

Summary. — Three cases of electron motion in an axially symmetric magnetic field displacing along the axis are discussed: 1) the magnetic field is displacing in such a manner that the electron orbit radius remains constant; 2) the magnetic field is displacing uniformly with the constant velocity $u < c$ and 3) the magnetic field is displacing with the light velocity c. The gained energy, the time and path of acceleration are calculated for each of the three cases. The results can be considered as a theoretical basis for linear betatrons.

1. – Introduction.

After a well-known assumption of Wideröe [1] only two types of the relativistic problem on the electron motion in alternating axially symmetrical magnetic fields were investigated in detail; the motion in a magnetic field parallel and uniform along the symmetry axis [2] and that in a barrel-shaped magnetic field having a minimum value at some fixed plane perpendicular to the symmetry axis [3,4]. We shall investigate a new type of problem, that of electron motion in alternating axially siymmetrical magnetic field when the mag-

(1) R. Wideröe: *Arch. f. El.*, **21**, 387 (1928).
(2) J. P. Terletskij: *Žu. Teor. Èksper. Fiz.*, **11**, 96 (1941).
(3) D. W. Kerst and R. Serber: *Phys. Rev.*, **60**, 53 (1941).
(4) J. P. Terletskij: *Journ. of. Phys. USSR*, **9**, 159 (1945).

netic field decreasing in the direction of the symmetry axis is displacing along the same axis with a certain velocity. As well as the former types, the new one can provide a theoretical basis for a new kind of accelerator, namely, for *linear induction accelerators* or, in short, *linear betatrons* ([5]). We shall consider three particular cases of this type of problem:

1) Orbit radius being constant in a field displacing non-uniformly;
2) Motion in a magnetic field displacing with velocity $u < c$;
3) Motion in a magnetic field displacing with the light velocity c.

2. – Equations of motion.

According to ([2-4]) the Lagrange function of the electron in an axially symmetric magnetic field is

$$\mathcal{L} = -m_0 c^2 \sqrt{1 - \frac{\dot{r}^2 + \dot{z}^2 + (r\dot{\varphi})^2}{c^2}} + \frac{e}{c} r\dot{\varphi} A(r, z, t) , \tag{1}$$

where A is the component of the vector potential A_φ and the other characters have the usual meaning. The field strength components are connected with the vector potential by the well-known expressions:

$$\begin{cases} H_z = \dfrac{1}{r} \dfrac{\partial}{\partial r} (rA) , \qquad H_r = -\dfrac{\partial A}{\partial z} , \qquad H_\varphi = 0 , \\ A = \dfrac{1}{2\pi r} \displaystyle\int_0^r H_z 2\pi r \, \mathrm{d}r + \frac{b}{r} = \frac{r\overline{H}}{2} + \frac{b}{r} , \end{cases} \tag{2}$$

where $\overline{H}$ is the mean value of z-component of the magnetic field inside a circle of radius r, b is the integration constant. By means of Lagrange equations we get:

$$\frac{\mathrm{d}}{\mathrm{d}t} (m\dot{r}) = \frac{e}{c} r\dot{\varphi} \frac{\partial A}{\partial r} = \frac{e}{c} r\dot{\varphi} \left[H_z - \frac{A}{r} \right] , \tag{3a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} (m\dot{z}) = \frac{e}{c} r\dot{\varphi} \frac{\partial A}{\partial z} = -\frac{e}{c} r\dot{\varphi} H_r , \tag{3b}$$

$$mr\dot{\varphi} = -\frac{e}{c} A , \tag{3c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} (mc^2) = -\frac{e}{c} r\dot{\varphi} \frac{\partial A}{\partial t} , \tag{3d}$$

([5]) M. V. Konjukov and J. P. Terletskij: *Žu. Èksper. Teor. Fiz.*, **34**, 1003 (1958).

where

$$m = \frac{m_0}{\sqrt{1 - \dfrac{\dot{r}^2 + \dot{z}^2 + (r\dot{\varphi})^2}{c^2}}} .$$

The integration constant of (3*c*) is chosen to be zero; however, this doesn't limit the generality of the equations since the choice of a certain value of this constant is the same as that of a certain gauge for the vector potential, *i.e.* of the constant b.

Putting $r\dot{\varphi}$ of (3*c*) into the equation for m we obtain:

$$m^2(\dot{r}^2 + \dot{z}^2 - c^2) + \frac{e^2}{c^2} A^2 + m_0^2 c^2 = 0 . \tag{4}$$

If the field is displacing along the axis with the constant velocity u, *i.e.* $A = A(r, z - ut)$, then according to (3*b*) and (3*d*) we have:

$$m\left(\dot{z} - \frac{c^2}{u}\right) = - M \frac{c^2}{u} , \tag{5}$$

where $M = m_0(1 - r_0^2\dot{\varphi}_0^2/c^2)^{-\frac{1}{2}}$ is the value of the mass at $\dot{z} = 0$ and $\dot{r} = 0$.

3. – Motion along a spiral of constant radius.

Let us consider the solution of the system of equations assuming the existence of a field $A = A(r, z, t)$ (*), which has $H_r = \text{const}$ and $H_z = \text{const}$ when $r = R = \text{const}$.

The requirement of motion in a circle of constant radius, according to (3*a*) and (2), gives:

$$H_z = \frac{A}{R} = \frac{\bar{H}}{2} + \frac{b}{R^2} . \tag{6}$$

From the equations (3*b*) and (3*c*) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{z})^2 = - \frac{e^2}{c^2} \dot{z} \frac{\partial A^2}{\partial z} = 2 \frac{e^2 R}{c^2} H_r H_z \dot{z} . \tag{7}$$

(*) The possibility of realization of such a field is discussed in Appendix A where the expression for the vector potential in the vicinity of the point $r = R$ and $z = z(t)$ is obtained.

Since H_r and H_z are constant this equation can be integrated. Considering that at the initial instant a particle being at $z = 0$ has the z-component of velocity equal to zero we obtain

$$(m\dot{z})^2 = 2\,\frac{e^2R}{c^2}\,H_rH_zz\,. \tag{8}$$

Assuming $\dot{r} = 0$ in (4) and expressing m in terms of $\dot{z}$ after substitution into (8) we obtain with the help of (6)

$$\dot{z} = \frac{c}{\sqrt{1 + m_0^2c^4\,\dfrac{1 + (eRH_z/m_0c^2)^2}{2e^2H_rH_zRz}}}\,. \tag{9}$$

Integrating (9) we have

$$ct = \int_0^z \mathrm{d}z\sqrt{1 + a/z} = \sqrt{z}\sqrt{z + a} + a\ln\left(\sqrt{z} + \sqrt{z + a}\right), \tag{10}$$

where

$$a = m_0^2c^4\,\frac{1 + (eRH_z/m_0c^2)^2}{2e^2RH_rH_z}\,.$$

The equation (10) of the guiding centre motion allows to find the displacement velocity of the constant magnetic field region where the discussed mode of the particle motion is ensured.

For the particle energy, according to (4), (6) and (8), we obtain

$$E^2 = (mc^2)^2 = 2e^2RH_rH_z(a + z)\,. \tag{11}$$

Let us discuss some limiting cases of (10) and (11). In the case of $z \gg a$ (*) (10) takes the form:

$$ct = z\,,$$

it means that the particle has a constant transversal velocity practically equal to the light velocity.

(*) In the case of $eRH_z/m_0c^2 \gg 1$, $a = RH_z/2H_r$ and when $H_r \simeq H_z$ the value of a has the order of about the orbit radius. In this case the assumption of $z \gg a$ means that the path along the z-axis is considerably larger than the orbit radius. It is valid when the energy of the perpendicular motion is larger than the electron rest energy or equal to it.

For large enough z and under the condition $e^2R^2H_z^2/m_0^2c^4 \gg 1$, (11) takes the form:

$$E \cong E_0 \sqrt{\frac{z}{R} \cdot \frac{2H_r}{H_z}} \,. \tag{13}$$

4. – Motion in a magnetic field displacing with constant velocity u.

Let us discuss at first the question of the magnitude of the energy gained by the particle in a magnetic field displacing with constant velocity. Let us suppose that the field displaces along the z-axis with the velocity $u < c$ and has a « bottle-shaped » form so that the « bottle neck region », *i.e.* the region where the field has his maximum value, is directed against the direction of the velocity u. Further, let $H_r = 0$ in the « broad part of the bottle » where the field has a minimum value, *i.e.* the field is parallel to the z-axis.

If originally the electron has $\dot{z} = 0$ when it is yet in the broad part of the « bottle », later entering the narrower part of the « bottle » it will be accelerated in direction of the z-axis and will attain the velocity u; after this it will be again pulled out into the broad part of the « bottle » acquiring the velocity u_2 larger than u. This is a picture of the motion which follows from the consideration in the system of reference moving with the field (see, for example, ([6])).

Now let us consider formula (4) in the next three moments:

a) $t = 0$; in the initial moment the electron is located in the broad part where $H_r = 0$.

Here we have

$$\dot{r} = 0\,, \qquad \dot{z} = 0\,, \qquad \text{and} \qquad m = M$$

and (4) takes the form

$$(m_0^2 - M^2)c^2 + \frac{e^2}{c^2} A_0^2 = 0\,, \tag{14a}$$

where A_0 is the vector potential in the broad part of the « bottle ».

b) $t = t_1$; in this moment the electron acquires the axial velocity u. It is evident that $\dot{r} = 0$ and $\dot{z} = 0$ and (4) takes the form:

$$m_1^2(u - c^2) + m_0^2c^2 + \frac{e^2}{c^2} A_1^2 = 0\,, \tag{14b}$$

([6]) E. FERMI: *Phys. Rev.*, **75**, 1169 (1949).

where m_1 is the mass of the electron at moment t and A_1 is the vector potential value in the point of the « bottle neck » where the electron is at rest relative to the field.

c) $t = t_2$; it is the moment of the electron's return into the broad part of the « bottle », where $H_r = 0$. Here is $\dot{r} = 0$, $\dot{z} = 0$, the mass of the electron is m_2, and (4) takes the form

$$m_2^2(u_2^2 - c^2) + m_0^2 c^2 + \frac{e^2}{c^2} A_0^2 = 0 . \tag{14c}$$

The elimination of A_0 from (14*a*) and (14*c*) gives

$$m_2^2(u_2^2 - c^2) + M^2 c^2 = 0 . \tag{15}$$

Since, according to (5),

$$m_2 \left(u_2 - \frac{c^2}{u} \right) + M \frac{c^2}{u} = 0 , \tag{16}$$

from (15) and (16) we obtain

$$u_2 = \frac{2u}{1 + u^2/c^2} , \tag{17}$$

$$m_2 = M \frac{1 + u^2/c^2}{1 - u^2/c^2} . \tag{18}$$

To determine the energy dependence on the maximal value of the magnetic field (in the « bottle neck ») we shall take advantage of the fact that, according to (5),

$$m_1 = \frac{M}{1 - u^2/c^2} \tag{19}$$

if we substitute (19) into (18) we shall obtain

$$m_2 = m_1(1 + u^2/c^2) . \tag{20}$$

If we substitute (19) into (14), then we shall have

$$\frac{1}{1 - u^2/c^2} = \left(\frac{m_0}{M} \right)^2 + \left(\frac{eA_1}{Mc^2} \right)^2 , \tag{21}$$

Then, according to (19) and (20)

$$\frac{m_2}{M} = 2 \left[\left(\frac{m_0}{M} \right)^2 + \left(\frac{eA_1}{Mc^2} \right)^2 \right] - 1 . \tag{22}$$

If the electron entering the broad part of the bottle has a relativistic initial energy ($M/m_0 \gg 1$), then (22) will take the more simple form:

$$\frac{m_0}{M} = 2\left(\frac{A_1}{A_0}\right)^2 - 1 . \tag{23}$$

If we assume that $r = r_0 = \text{const}$ in the broad part of the magnetic « bottle », then, according to (3a), $A_0 = H_0 r_0$ and, according to (2) $b = r_0^2(H_0 - \overline{H}_0/2)$. Hence, $A_1 = r_1\overline{H}_1/2 + (H_0 - \overline{H}_0/2)r_0^2/r_1$, where H_0, H_1, and $\overline{H}_0$, $\overline{H}_1$ are the values of H_z and $\overline{H}$ in the points where r is equal to r_0 and r_1 respectively. Under the slow enough change of the magnetic field with the change of z we may consider $H_z \simeq \overline{H}$ and, according to the adiabatic invariant conservation, $H_z r^2 = \text{const}$. Hence, in this case $A_1 = r_1 H_1 = r_0\sqrt{H_0 H_1}$ and then we obtain

$$\frac{E}{E_0} = \frac{m_0}{M} = 2\frac{H}{H_0} - 1 . \tag{23a}$$

Now let us discuss the problem of the time necessary to accelerate the particle to the energy given by the formulae (23), (23a) and on the path which the particle covers in direction of the z-axis during this time. It is simpler to consider this problem in a system of reference moving with velocity u, *i.e.* together with the magnetic field, since in this case the kinetic energy doesn't change, *i.e.*, $m = \text{const}$.

With the help of considerations similar to those used to obtain the result (23a) and assuming $\dot{r} \ll \dot{z}$ we obtain from the formula (4) that

$$\frac{m\dot{z}^2}{2} = -\mu\,[H(z) - H_m] , \tag{24}$$

where $\mu = e^2 H r^2/2mc^2$ is the magnetic moment and H_m is the magnetic field strength in the point $z = z_m$, where the particle's axial velocity is zero. The second integration of (24) gives the following magnitude of the motion time interval from the initial point z_0, where the magnetic field value is H_0, to the point z_m:

$$T = \frac{1}{v_{\perp 0}}\int_{z_0}^{z} \frac{\mathrm{d}z}{\sqrt{\dfrac{H_m - H(z)}{H_0}}} , \tag{25}$$

where $v_{\perp 0}$ is the initial transversal velocity. To evaluate the magnitude of T we assume that the magnetic field in the region of motion of the charged particle is

$$H(z) = H_0 + (H_m - H_0)\frac{z - z_0}{z_m - z_0} , \tag{26}$$

and the initial velocity and the field values H_m and H_0 are connected by

$$H_m = H_0\left[1+\left(\frac{v_{z_0}}{v_{\perp_0}}\right)^2\right] \quad (*) \tag{27}$$

then with the help of simple calculations we obtain the explicit expression of the acceleration time:

$$2T = \frac{4(z_m - z_0)}{v_{z_0}}\,, \tag{28}$$

where $v_{z_0} = u$ is the limit of the initial velocity of charged particles.

At the transition to the system of reference where in the initial moment the particle is at rest, the time of motion in the interval $z_m - z_0$ takes the form:

$$T' = \frac{2(z_m - z_0)}{u\sqrt{1-u^2/c^2}} \tag{29}$$

and the interval length $(z - z_0)$ becomes

$$l = (z_m - z_0)\left(1 - \frac{u^2}{c^2}\right). \tag{30}$$

The calculation of the acceleration path in the system relative to which the magnetic field is displacing with the velocity u, gives

$$L = 2T'u = \frac{4l}{1-(u^2/c^2)} \cong 2l\frac{E}{E_0}\,. \tag{31}$$

5. – Motion in a magnetic field displacing with the light velocity.

Let us suppose that differently from Sect. 4, the magnetic field is displacing with the light velocity, *i.e.* $A = A(r, z - ct)$. Then with the help of simple transformations, we obtain (3*a*) and (3*d*) in the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{z})^2 = -\frac{e^2}{c^2}\,\dot{z}\,\frac{\partial A^2}{\partial z} \tag{32}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}(mc^2)^2 = \frac{e^2}{c^2}\,\frac{\partial A^2}{\partial t} = -\frac{e^2}{c^2}\,c\,\frac{\partial A^2}{\partial z}\,. \tag{33}$$

(*) (27) follows from (24) under the assumption that the adiabatic invariant is conserved during the motion.

When the field is displacing with the light velocity, with the help of the equations of electrodynamics we obtain

$$A = \frac{H(\zeta)r}{2} + \frac{b}{r}, \tag{34}$$

where $H(\zeta) = H_z$ is an arbitrary function of $\zeta = z - ct$ and b is a constant (see Appendix B). Substituting (34) into (32) and (33) and subtracting the second expression from the first one we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(m^2\dot{z}^2 - m^2c^2) = -\frac{e^2}{c^2}\left[\frac{Hr^2}{2} + b\right]\frac{\mathrm{d}H}{\mathrm{d}t}. \tag{35}$$

Using the assumption of the adiabatic invariant conservation ($Hr^2 = \text{const}$) and supposing that, as in the previous case, $b = H_0 r_0^2/2$, after the integration of (35) we obtain

$$m^2(\dot{z}^2 - c^2) + \frac{e^2}{c^2}(Hr^2)[H - H_0] = -M^2c^2. \tag{36}$$

Then we use formula (5) that at $u = c$ takes the more simple form:

$$m(\dot{z} - c) = -Mc. \tag{37}$$

The elimination of m from (36) and (37) and the use of the variable $\zeta = z - ct$ give

$$\frac{2c}{\dot{\zeta}} = -\frac{e^2}{M^2c^4}(Hr^2)[H - H_0] - 2, \tag{38}$$

and the integration of (38) gives the equation of the guiding centre motion:

$$z - z_0 = -\frac{e^2(Hr^2)}{2M^2c^4}\int_{z_0}^{z-ct}(H - H_0)\,\mathrm{d}\zeta. \tag{39}$$

According to (37) and (38) the energy change during the acceleration time is:

$$(m - M)c^2 = \frac{e^2}{2Mc^2}(H_0 r_0^2)[H - H_0]. \tag{40}$$

When the initial energies are relativistic ones, according to (4), we have $Mc = -(e/c)A_0$, and since $A_0 = H_0 r_0$, we obtain

$$\frac{E}{E_0} - 1 = \frac{m}{M} - 1 \simeq \frac{1}{2}\left(\frac{H}{H_0} - 1\right). \tag{41}$$

If we take (26) for $H(\zeta)$, then, according to (39) and (41), we shall obtain for the acceleration path

$$L/l = E/E_0 \,, \tag{39a}$$

where l is the length of the increasing field region.

We can see from (41) that the energy of the charged particle which moves in such a magnetic field displacing with the light velocity increases as about H/H_0.

Appendix A

Field in the vicinity of an orbit of constant radius moving with variable velocity.

The particle motion along a spiral of constant radius is effected in the magnetic field displacing with velocity $\dot{z}$ in direction of the positive z axis. To determine the field it is sufficient to know the φ-component of the vector potential A satisfying the equation,

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA)\right] + \frac{\partial^2 A}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0 \,. \tag{42}$$

We assume that the dependence of the field on r in the vicinity of the moving particle has for the betatron such a characteristic form that

$$\left.\frac{\partial H_z}{\partial r}\right|_{\substack{r=R \\ z=z(t)}} < 0 \,, \tag{43}$$

but the dependance on z differs from that of betatron and is supposed to satisfy the condition

$$\left.\frac{\partial A}{\partial z}\right|_{\substack{r=R \\ z=z(t)}} < 0 \,. \tag{44}$$

Considering that the time dependence of A has the form $Z(r, z-z(t), t)$ and changing the variables r, z, t into r, $\zeta = z - z(t)$, t we obtain the following form of the equation for A:

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA)\right] + \left[1 - \frac{\dot{z}^2(t)}{c^2}\right]\frac{\partial^2 A}{\partial \zeta^2} + \frac{\ddot{z}(t)}{c^2}\frac{\partial A}{\partial \zeta} + \frac{2\dot{z}(t)}{c^2}\frac{\partial^2 A}{\partial t\,\partial \zeta} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 0 \,. \tag{45}$$

As in the case of betatron [4] we shall look for the solution of (45) in the form:

$$A = r(a_0 + a_1 r + a_2 r^2 + b_1 \zeta + b_2 \zeta^2 + a_{12} r\zeta)\,. \tag{46}$$

If we may suppose $\ddot{z} \ll 8a_2 c^2/b_1$, then the solution in the vicinity of the point $\zeta = 0$, $r = R$ with the accuracy of the second order infinitesimal relative to $\varrho = r - R$ and $\zeta = z - z(t)$ will have the form:

$$A = (R + \varrho)\left[\frac{H_z^0}{2} - \frac{a}{2} R^2 + \frac{a}{4}(R + \varrho)^2 - \frac{H_r^0}{R}\zeta - \frac{a\zeta^2}{1 - (\dot{z}^2/c^2)}\right], \tag{47}$$

where $a = 4a_2$, H_z^0 and H_r^0 are the values of H_z and H_r at $\varrho = \zeta = 0$, $b_1 = -H_r^0/R$.

Putting (47) into (2) we obtain the constants $H_r = H_r^0$ and $H_z = H_z^0$ at the line $\varrho = 0$, $\zeta = 0$. Thus for not very large accelerations $\ddot{z}$ ($\ddot{z} \ll c^2(\partial H_r/\partial r)_0/H_r^0$), $H_r = \text{const}$ and $H_z = \text{const}$ at the equilibrium orbit line.

Appendix B

Vector potential in the case of a magnetic field displacing with the light velocity.

Since in the present case $A = A(r, z - ct)$ the equation of the vector potential has the form:

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA)\right] = 0 \tag{48}$$

the solution of (48) is

$$A = \frac{ar}{2} + \frac{b}{r}\,, \tag{49}$$

where a and b, generally speaking, are arbitrary functions of $\zeta = z - ct$. With the use of (2) we obtain that $a = H_z(\zeta)$, and to satisfy the condition of the finiteness of H_r at the symmetry axis of the field we must choose the constant b. Definitely for the field vector potential we have the following expression:

$$A = \frac{rH(\zeta)}{2} + \frac{b}{r}\,. \tag{50}$$

6. – Conclusion.

We have discussed three cases of motion of a charged particle in a displacing magnetic field. The energy gained by the particle in the non-uniformly moving magnetic field is determined both by the field strength and the particle path.

In the two other cases the energy increase can be evaluated with the help of the following expressions:

$$E/E_0 \simeq 2(H/H_0) \qquad \text{or} \qquad E/E_0 \simeq \tfrac{1}{2}(H/H_0)\,,$$

which are similar to the ones of betatron. However, we must note that at the acceleration by the displacing magnetic field the strong field can be concentrated in a considerably smaller region, since $Hr^2 = \text{const}$. We didn't investigate the problem of the motion stability, and it is necessary to examine it separately.

RIASSUNTO (*)

Si discutono tre casi di moto degli elettroni in un campo magnetico a simmetria assiale che si sposti in direzione dell'asse: 1) il campo magnetico si sposta in modo che il raggio dell'orbita dell'elettrone si mantenga costante; 2) il campo magnetico si sposta con moto uniforme e velocità $u < c$ costante; 3) il campo magnetico si sposta con la velocità c della luce. Per ciascuno dei tre casi si calcola il guadagno d'energia, il tempo e il percorso d'accelerazione.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

On a Possible Geometrical Interpretation of Gauge Trasformations (*).

J. Rzewuski

Institute of Physics of the Polish Academy of Sciences - Wrocław
Institute of Theoretical Physical, University of Wrocław - Wrocław

(ricevuto il 27 marzo 1958)

Summary. — It is shown that the geometrical origin of gauge transformations may be found if one considers the eight-dimensional spinor space rather than the four-dimensional vector space as the basis of a physical theory. The connexion of the spinor space with the vector space is studied in some detail. It is shown that the irreducible representations of an enlarged group of transformations of the spinor space contain all the transformation types assumed for fundamental particles, including their isotopic spin characterization.

Introduction.

It is well known that the laws of conservation of energy-momentum and angular momentum result from the invariance of isolated systems with respect to translations and rotations of space-time. In other words these laws express the homogeneity and isotropy of space-time. The other important conservation law—the charge conservation law—possesses no geometrical explanation. This difference in the origin of the notions of charge on the one side and of energy-momentum and angular momentum on the other is unsatisfactory and indicates that our conventional four-dimensional geometry is too superficial or too narrow to provide a geometrical basis for all conservation laws in physics.

Therefore it is the purpose of this paper to go back to the basic notion of the Lorentz group namely to the notion of a spinor and consider the spinor

(*) A preliminary report of some of the results of this paper was published in the *Bulletin de l'Académie Polonaise des Sciences, Cl. III*, No. 4, (1958) (quoted hereafter as (I)) and No. 5 (1958) (quoted hereafter as (II)).

space rather than the four-dimensional vector space of Minkowski as the basis for a physical theory. The Minkowski space appears then as a secondary notion. In frames of this space it is easy to overlook certain transformations of the spinor space which lead us directly to the charge-conservation law and which may be brought into connexion with isotopic spin transformations. Besides, the representations corresponding to the enlarged group of transformations seem to be useful for the classification of fields occurring in nature from a unified point of view.

1. – The spinor space.

Let us consider the spinor space consisting of the four complex (eight real) variables z_α, $z_{\dot\alpha}$ ($\alpha = 1, 2$). This space is usually defined by the following transformation law for the variables:

$$\begin{cases} z'_1 = \alpha z_1 + \beta z_2 , & z^{*\prime}_{\dot 1} = \alpha z^*_{\dot 1} + \beta z^*_{\dot 2} , \\ z'_2 = \gamma z_1 + \delta z_2 , & z^{*\prime}_{\dot 2} = \gamma z^*_{\dot 1} + \delta z^*_{\dot 2} , \end{cases} \qquad \alpha\delta - \beta\gamma = 1, \tag{1.1}$$

where α, β, γ, δ are arbitrary complex numbers satisfying condition (1.1). These transformations form the socalled $\boldsymbol{c}_2$-group. From the point of view of transformations (1.1) it is possible to consider the dotted variables as complex conjugate to the undotted. However, already the necessity to represent inversions of time by transformations of first rank spinors makes any connexion between the dotted and the undotted quantities impossible. We shall see in the following another reason for considering the dotted and the undotted variables as independent quantities.

Of course, it is possible to consider far more general transformations of the variables z (z_α, $z_{\dot\alpha}$) than those expressed by (1.1). In this paper we shall concentrate upon only one group of such transformations, namely upon the group (*)

$$\begin{cases} z'_\alpha = a z_\alpha + b z^*_{\dot\alpha} , \\ z^{*\prime}_{\dot\alpha} = c z_\alpha + d z^*_{\dot\alpha} , \end{cases} \qquad ad - bc = 1, \qquad (\alpha = 1, 2). \tag{1.2}$$

(*) The unitary sub-group $\boldsymbol{u}'_2$ of this group was considered by PAULI (*Nuovo Cimento* **6**, 204 (1957)) and GÜRSEY (*Nuovo Cimento*, **7**, 411 (1958)) in another connexion. Apart from the fact that these authors consider a sub-group only, there are two other essential differences: 1) PAULI and GÜRSEY consider the transformations of $\boldsymbol{u}'_2$ as additional transformations on the fields, the latter being representations on the $\boldsymbol{c}_2$-group. In this paper the fields are considered to be representations of the full product group $\boldsymbol{c}_2 \cdot \boldsymbol{c}'_2$ (cf. (1.3) and Sect. **4**.) 2) In the formulation of PAULI and GÜRSEY the transformations of the $\boldsymbol{u}'_2$ do not affect the variables X_μ of the Minkowski space whereas in our approach these variables are functions of the spinor variables and, therefore, the transformations of $\boldsymbol{c}'_2$ induce, in general transformations of the X_μ (cf. Sect. **2** and **3**).

We shall call this group of transformations the $\boldsymbol{c}_2'$-group. The coefficients of (1.2) are, of course, considered to be independent of the coefficients of (1.1).

Amongst all the transformations of the variables z, the transformations (1.2) are particularly interesting due to the fact that they commute with the transformations (1.1). We may, therefore, define the direct product of the groups $\boldsymbol{c}_2$ and $\boldsymbol{c}_2'$

$$\boldsymbol{c}_2 \cdot \boldsymbol{c}_2' = \boldsymbol{c}_2' \cdot \boldsymbol{c}_2 \,. \tag{1.3}$$

There is, *a priori*, no reason for preferring any of them. They appear quite symmetrically as possible transformation groups of the spinor space.

We notice that the transformations $\boldsymbol{c}_2$ and $\boldsymbol{c}_2'$ change their rôles if we carry out one of the following substitutions:

$$\begin{array}{ll} z_2 \to z_{\dot 1}^* & z_{\dot 2} \to z_2 \\ z_1 \to z_1 & z_{\dot 1}^* \to z_2 \end{array} \quad \text{or} \quad \begin{array}{ll} z_2 \to z_2 & z_{\dot 2}^* \to z_1 \\ z_1 \to z_{\dot 2}^* & z_{\dot 1} \to z_{\dot 1} \,. \end{array} \tag{1.4}$$

2. – The connexion between the spinor space and the Minkowski space.

It is well known that the connexion between vectors X_μ of a vector space with Minkowski's metric and the components $z_{\alpha\dot\beta}$ of a second rank spinor which transform according to transformations of the $\boldsymbol{c}_2$-group are of the form

$$\begin{aligned} z_{1\dot 1} &= X_3 - X_0 \qquad z_{1\dot 2} = X_1 - iX_2 \,, \\ z_{2\dot 1} &= X_1 + iX_2 \qquad z_{2\dot 2} = -X_3 - X_0 \,. \end{aligned} \tag{2.1}$$

Now if we consider the X_μ to be co-ordinates of a point in Minkowski's space we obtain the most general connexion of these co-ordinates with the variables z by assuming the $z_{\alpha\dot\beta}$ to be functions of these variables

$$z_{\alpha\dot\beta} = z_{\alpha\dot\beta}(z_\lambda, z_{\dot\lambda}, z_\lambda^*, z_{\dot\lambda}^*) \,. \tag{2.2}$$

For any detailed investigation it is, of course, necessary to fix the functional dependence (2.2) in some way. We shall assume in the following this functional dependence to be bilinear.

It was shown in I) that the most general bilinear form transforming like $z_{\alpha\dot\beta}$

with respect to c_2 and satisfying the conditions $z_{1\dot{1}} = z^*_{1\dot{1}}$, $z_{1\dot{2}} = z^*_{2\dot{1}}$, $z_{2\dot{2}} = z^*_{2\dot{2}}$, is

$$z_{\alpha\dot{\beta}} = \mu z_\alpha z^*_{\dot{\beta}} + \nu z^*_{\dot{\alpha}} z_{\dot{\beta}} \,, \tag{2.3}$$

where μ and ν are two arbitrary real parameters (*).

For the connexion of the X_ν with the z one obtains by means of (2.3) and (2.1) explicitly

$$\left\{\begin{aligned} X_1 &= \frac{\mu}{2}(z_1 z^*_2 + z_2 z^*_1) + \frac{\nu}{2}(z^*_{\dot{1}} z_{\dot{2}} + z^*_{\dot{2}} z_{\dot{1}}) \,, \\ X_2 &= i\frac{\mu}{2}(z_1 z^*_2 - z_2 z^*_1) + i\frac{\nu}{2}(z^*_{\dot{1}} z_{\dot{2}} - z^*_{\dot{2}} z_{\dot{1}}) \,, \\ X_3 &= \frac{\mu}{2}(z_1 z^*_1 - z_2 z^*_2) + \frac{\nu}{2}(z^*_{\dot{1}} z_{\dot{1}} - z^*_{\dot{2}} z_{\dot{2}}) \,, \\ -X_0 &= \frac{\mu}{2}(z_1 z^*_1 + z_2 z^*_2) + \frac{\nu}{2}(z^*_{\dot{1}} z_{\dot{1}} + z^*_{\dot{2}} z_{\dot{2}}) \,. \end{aligned}\right. \tag{2.4}$$

The scalar product of the vector X_μ with another vector Y_μ is easily seen tob e

$$X_\mu Y_\mu = -\tfrac{1}{2}\{\mu\mu' |z_\alpha u^\alpha|^2 + \nu\nu' |z^*_{\dot{\alpha}} u^{\dot{\alpha}*}|^2 + \mu\nu' |z_\alpha u^{\dot{\alpha}*}|^2 + \mu'\nu |z^*_{\dot{\alpha}} u^\alpha|^2\} . \tag{2.5}$$

For $z = u$ and $\mu = \mu'$, $\nu = \nu'$ we get $X_\mu = Y_\mu$ and

$$X^2_\mu = -\mu\nu |z_\alpha z^{\dot{\alpha}*}|^2 . \tag{2.6}$$

It is easily seen from (2.6) that we get points outside of the light-cone if $\mu\nu < 0$ and inside of the light-cone if $\mu\nu > 0$. From the last equation (2.4) it is seen further that the past light cone corresponds to $\mu > 0$, $\nu > 0$ and the future light cone to $\mu < 0$, $\nu < 0$. Thus the values of the parameters μ and ν or of their product $\mu\nu$ determine uniquely to which of the three domains of space-time a point X_μ belongs. A detailed discussion of these circumstances was given in (I). Here we want only to remark that we could fix the parameters μ and ν, *e.g.* in the following way:

$$\left\{\begin{aligned} &\mu = 1 \,, \quad \nu = 1 && \text{for the past light cone,} \\ &\mu = -1 \,, \quad \nu = -1 && \text{for the future light cone,} \\ \text{or} \quad &\mu = 1 \,, \quad \nu = -1 \\ &\mu = -1 \,, \quad \nu = 1 && \text{for the outside of the light cone,} \end{aligned}\right. \tag{2.7}$$

(*) The form (2.3) is obtained from the form used in (I) by the substitution $z_\alpha \to \frac{1}{2}(z_\alpha - z^*_{\dot{\alpha}})$, $z^*_\alpha \to \frac{1}{2}(z_\alpha + z^*_{\dot{\alpha}})$. The parameters $\varkappa$ and λ of (I) are connected with μ and ν by means of the equations $\mu = \frac{1}{2}(\varkappa + \lambda)$, $\nu = \frac{1}{2}(\varkappa - \lambda)$.

all points of one domain being obtained by proper movements of the spinor space. However, there is no harm in keeping μ and ν in (2.4) undetermined and we shall do so in the following calculations.

Another point which we would like to emphazise is that by putting $z_\alpha = \varrho z^*_\alpha$ (ϱ an arbitrary complex number) we get $z_\alpha z^{\dot\alpha *} = \varrho^* z_\alpha z^\alpha = 0$ and, therefore, $X^2_\mu = 0$. By means of a two-dimensional complex spinor space we obtain only the particular case of a geometry on the light cone.

3. – Gauge transformations.

It is easily seen from (2.6) that the length of a vector in the X-space is invariant with respect to transformations of the full $\boldsymbol{c}_2 \cdot \boldsymbol{c}_2'$-group. Let us search for those transformations of this group which leave also the scalar product (2.5) unchanged. This scalar product is certainly invariant with respect to transformations of the $\boldsymbol{c}_2$-group.

To infer its behaviour with respect to transformations of $\boldsymbol{c}_2'$ we recall the well known fact that each transformation of this group may be written as a product of a certain finite number of three typical transformations

(3.1) 1) $\quad z'_\alpha = \exp[i\vartheta] z_\alpha , \qquad z^{*\prime}_{\dot\alpha} = \exp[-i\vartheta] z^*_{\dot\alpha} ,$

(3.2) 2)
$$\begin{cases} z'_\alpha = z_\alpha \cos\varphi + i z^*_{\dot\alpha} \sin\varphi , \\ z^{*\prime}_{\dot\alpha} = i z_\alpha \sin\varphi + z^*_{\dot\alpha} \cos\varphi , \end{cases}$$

(3.3) 3)
$$\begin{cases} z'_\alpha = \tfrac{1}{2}(l^{-1} + l) z_\alpha + \tfrac{1}{2}(l^{-1} - l) z^*_{\dot\alpha} , \\ z^{*\prime}_{\dot\alpha} = \tfrac{1}{2}(l^{-1} - l) z_\alpha + \tfrac{1}{2}(l^{-1} + l) z^*_{\dot\alpha} . \end{cases}$$

Now it is easily seen that the transformations (3.1) leave the co-ordinates X_μ and, therefore, also the scalar product (2.5) unchanged. These transformations form a commutative subgroup of $\boldsymbol{c}_2'$. We shall call this subgroup $\boldsymbol{g}_2'$.

It may be easily proved that the other two transformations (3.2) and (3.3) change in general the co-ordinates X_μ as well as the scalar product (2.5). We shall come back to this question presently.

The direct product of the groups $\boldsymbol{c}_2$ and $\boldsymbol{g}_2'$ consists thus of transformations which do not change the scalar product (2.5). This is more than the conventional Lorentz group. The admissible physical theories (describing isolated systems) must, of course, be invariant with respect to transformations of the full product group $\boldsymbol{g}_2' \cdot \boldsymbol{c}_2$.

One of the simplest irreducible representations of the group $\boldsymbol{c}_2 \cdot \boldsymbol{c}_2'$ are the first rank spinors ψ_α, $\psi_{\dot\alpha}$ which transform with respect to transformations of $\boldsymbol{c}_1 \cdot \boldsymbol{c}_2'$ like the co-ordinates z_α, $z_{\dot\alpha}$. Transformations of $\boldsymbol{c}_2$ are rotations of the X_μ-space, transformations of $\boldsymbol{g}_2'$ (cf. (3.1))

$$\psi'_\alpha(z') = \exp[i\vartheta]\psi_\alpha(z)\,, \qquad \psi'_{\dot\alpha}(z') = \exp[i\vartheta]\psi_{\dot\alpha}(z) \tag{3.5}$$

are gauge transformations of the first kind. The invariance of physical laws with respect to these transformations leads directly to the conservation law of charge. Quantization of rotations corresponding to (3.1) leads automatically to the appearance of a quantum of charge. It may be emphasized that (3.1) are the only linear transformations with the property to leave the co-ordinates X_μ unchanged (*).

It is interesting to note that there exists a larger manifold of transformations possessing the above property. An example is provided by the non-linear transformation

$$z'_\alpha = \exp[i\vartheta(z)]z_\alpha\,, \qquad z'_{\dot\alpha} = \exp[i\vartheta(z)]z_{\dot\alpha}\,, \tag{3.6}$$

representing gauge transformations of the second kind.

To obtain a general survey of such non-linear transformations let us remark that the four co-ordinates X_μ are expressed as four bilinear forms in the eight real variables $\mathrm{Re}\, z_\alpha$, $\mathrm{Re}\, z_{\dot\alpha}$, $\mathrm{Im}\, z_\alpha$, $\mathrm{Im}\, z_{\dot\alpha}$ ($\alpha = 1, 2$) (cf. also the considerations of (I), in particular formula (I.5.5)).

If we consider the X_μ as constant we obtain in (2.4) four equations for four second order hypersurfaces in the eight-dimensional real spinor space. The cut of these four surfaces represents in general a four-dimensional space. Any movement in this space leaves the X_μ unchanged. It is tempting to connect these movements with some internal structure of the fundamental particles.

Let us investigate now the effect of the other transformations of the $\boldsymbol{c}_2'$-group on the co-ordinates X_μ. For this purpose we introduce (3.2) and (3.3)

(*) This is so as long as we consider the group $\boldsymbol{c}_2 \cdot \boldsymbol{c}_2'$ only. If we put upon the transformation coefficients the less stringent condition $\alpha\delta - \beta\gamma = \exp[i\varphi]$ we get instead of $\boldsymbol{c}_1 \cdot \boldsymbol{c}_2'$ a 13-parametric group which is the direct product of $\boldsymbol{c}_2$, $\boldsymbol{c}_2'$ and the commutative group (cfr. (I))

$$z'_\alpha = \exp[i\vartheta]z_\alpha\,, \qquad z'_{\dot\alpha} = \exp[-i\vartheta]z_{\dot\alpha}\,.$$

It is easily seen from (2.3) that these transformations also do not change the co-ordinates X_μ. It was shown by Pauli and Gürsey (see references, page 943) that they are connected with the conservation law of fermions.

into (2.3) and obtain respectively

$$z'_{\alpha\dot\beta} = \mu \cos^2\varphi + \nu \sin^2\varphi) z_\alpha z^*_\beta + (\mu \sin^2\varphi + \nu \cos^2\varphi) z^*_{\dot\alpha} z_{\dot\beta} + \\ + i(\nu - \mu)(z_\alpha z_{\dot\beta} - z^*_{\dot\alpha} z^*_\beta) \sin\varphi \cos\varphi \,, \tag{3.7}$$

and

$$z'_{\alpha\dot\beta} = \tfrac{1}{4}[\mu(l^{-1}+l)^2 + \nu(l^{-1}-l)^2] z_\alpha z^*_\beta + \tfrac{1}{4}[\mu(l^{-1}-l)^2 + \nu(l^{-1}+l)^2] z^*_{\dot\alpha} z_{\dot\beta} + \\ + \tfrac{1}{4}(\mu+\nu)(l^{-2}-l^2)(z_\alpha z_{\dot\beta} + z^*_{\dot\alpha} z^*_\beta) \,. \tag{3.8}$$

It is seen from (3.7-8) that invariance with respect to transformations (3.2) may be obtained by putting $\mu = \nu$ whereas invariance with respect to (3.3) by putting $\mu = -\nu$. Thus for time-like vectors ($\mu = \nu$) we have invariance with respect to (3.1) and (3.2). These transformations form together the unitary subgroup $\boldsymbol{u}'_2$ of $\boldsymbol{c}'_2$. On the other hand for space-like vectors ($\mu = -\nu$) we have invariance with respect to (3.1) and (3.3). These transformations form together another subgroup of $\boldsymbol{c}'_2$ corresponding to rotations in a three-dimensional space with indefinite metric.

4. – Final remarks.

It is tempting to bring the transformations induced by the $\boldsymbol{c}'_2$-group into connexion with isotopic spin transformations. For this purpose it is necessary to know all irreducible representations of the direct product $\boldsymbol{c}_2 \cdot \boldsymbol{c}'_2$. Using the method of infinitesimal transformations one easily finds the 12 operators I_i ($i = 1, ..., 12$) generating these representations. They may be divided into four groups I^i_2, I^i_+, I^i_- ($i = 1, ..., 4$) in such a way that the operators belonging to different groups commute with each other. Inside each group they satisfy the commutation relations

$$\begin{cases} [I^i_2, I^i_+] = I^i_+ \\ [I^i_2, I^i_-] = -I^i_- \\ [I'_+, I'_-] = 2I^i_2 \,. \end{cases} \qquad (i = 1, ..., 4). \tag{4.1}$$

These commutation relations determine uniquely the representations and it is easily seen that among the irreducible representations of $\boldsymbol{c}_2 \cdot \boldsymbol{c}'_2$ all transformation types assumed for fundamental particles occur including their isotopic spin characterization.

It should be emphasized that the transformations of the $\boldsymbol{c}'_2$-group induce in general (cf. Sect. 3) certain transformations of the variables X_μ of the Minkowski space. If one takes the spinor space as the geometrical basis of

the theory, one has to consider differential equations in this space rather than in the X_μ-space. Some results concerning the connexion in the spinor space of the variables z_α, $z_{\dot\alpha}$ with the first and second order differential equations in the X_μ-space were published in (II). It is shown there that the equations in the spinor space possess a larger manifold of solutions than the conventional equations in the X_μ-space. Only a part of this manifold coincides with the solutions of the Dirac or Klein-Gordon equations respectively. This situation is a necessary consequence of the point of view adopted in this paper according to which the transformations of the $\boldsymbol{c}_2'$-group are put on equal footing with the transformations of the $\boldsymbol{c}_2$-group as transformations of the same geometrical space, the spinor space.

RIASSUNTO (*)

Si dimostra che si può trovare l'origine geometrica delle trasformazioni di gauge prendendo a base di una teoria fisica lo spazio spinoriale a otto dimensioni anzichè lo spazio vettoriale quadridimensionale. Si studia nel dettaglio la connessione dello spazio spinoriale con lo spazio vettoriale. Si dimostra che le rappresentazioni irriducibili di un gruppo allargato di trasformazioni dello spazio spinoriale contiene tutti i tipi di trasformazioni assunti per le particelle fondamentali, ivi inclusa la caratterizzazione del loro spin isotopico.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

The Nucleon Effective Mass and the Statistical Model of the Nucleus.

E. Clementel (*) and C. Villi (+)

(*) *Istituto di Fisica dell'Università - Ferrara*
(*) *Istituto Nazionale di Fisica Nucleare - Sezione di Padova*
(+) *Istituto di Fisica dell'Università - Trieste*
(+) *Istituto Nazionale di Fisica Nucleare - Sottosezione di Trieste*

(ricevuto il 21 Maggio 1958)

Summary. – The statistical model of the nucleus has been refined taking into account the momentum dependence of the real part of the nucleon-nucleus potential. The promising features of this model are discussed with special regard to the concept of the nucleon effective mass. The determination of the nucleon effective mass is carried out taking into account separately the experimental value of the symmetry energy and of the nuclear level density observed in slow neutron capture experiments.

Introduction.

This paper is an attempt to study along general lines the momentum dependence of the nucleon-nucleus potential inside the Fermi sphere. It is proved in Sect. 1·1 that the momentum dependence varying linearly with the square of the nucleon momentum does not depend, in the first order approximation, on the preconceived choice of the two body potential. The nucleon effective mass, following from the velocity dependence of the potential, has been also derived on the basis of the correspondence principle (Sect. 1·2). The value of the nucleon effective mass, evaluated according to elementary hydrodinamical considerations (Sect. 1·3), is found equal to half the free nucleon mass, in agreement with other determinations. The radial and momentum dependence of the nucleon-nucleus potential, derived in the above mentioned approximation, is discussed in Sects. 1·4 and 1·5. The influence of the momentum dependence of the real part of the potential on the imaginary one is briefly discussed in Sect. 1·6.

The nucleon effective mass concept is extensively applied to the statistical model of the nucleus, and a modified expression of the nuclear excitation energy is derived in Sect. 2·1. Such a modification is shown to follow also from the explicit introduction of two body forces into the model (Sect. 2·2). Some features of the considered modified statistical model are briefly discussed in Sect. 2·3, with reference to Bardeen's correlated model and Watanabe's theory. The nucleon effective mass is determined in Sect. 2·4 from the experimental value of the symmetry energy, and in Sect. 2·5 from the level density measured in slow neutron capture experiments. The effect of the energy dependence of the potential on the evaporative spectra is discussed in Sect. 2·6. The results of this paper are only preliminary, and the main object of the present investigation is simply to show that several otherwise conflicting aspects of the nucleus theory can be accounted for on the basis of the nucleon effective mass concept.

1. – The nucleon effective mass approximation. The real and imaginary part of the nucleon-nucleus potential.

1·1. *The real part of the nuclear potential. The momentum dependence of the potential inside the Fermi sphere.* – The statistical model of the nucleus is based on the assumption that a heavy nucleus is formed by two non-interacting strongly degenerated gases of Z protons and $N = A - Z$ neutrons, moving freely within a spherical attractive potential well of radius $R = r_0 A^{\frac{1}{3}}$, with depth $\mathcal{V}_0$ adjusted so that the Fermi energy E_F raises the highest lying nucleon up to the observed binding energy B. The total energy of a nucleon bound in the nucleus is therefore assumed to depend on the nucleon momentum (*) k through the kinetic energy alone, *i.e.*

$$\mathcal{E}(k) = \frac{k^2}{2M} + \mathcal{V}_0 \,. \tag{1}$$

The momentum dependence of $\mathcal{E}(k)$, given in Eq. (1), is conflicting with the elastic scattering experiments of nucleons by nuclei. The Taylor [1] analysis has indeed shown that the real part $\mathcal{V}_R$ of the average potential, encountered by a nucleon inside the nucleus, varies with the nucleon energy E. A satisfactory agreement between the calculated cross-sections and the observed

(*) A system of units where $\hbar = c = 1$ ($\mu^{-1} = 1.4 \cdot 10^{-13}$ cm) is used.

(1) T. B. Taylor: *Phys. Rev.*, **92**, 831 (1953); *Nuclear scattering of high energy neutrons and the optical model* (unpublished).

ones in the energy range (60÷300) MeV is obtained provided $|\mathcal{V}_R(E)|$ behaves as a monotonically decreasing function of the neutron energy from $\mathcal{V}_R = -30$ MeV for $E \simeq 50$ MeV to $\mathcal{V}_R = -10$ MeV for $E \simeq 300$ MeV (Fig. 1, curve *a*). Neglecting, as a first approximation, the dependence of the potential on the radial distance r from the nucleus center (Sect. 1·4), this experimental result suggests that Eq. (1) should be replaced by

$$\mathcal{E}(k) = \frac{k^2}{2M} + \mathcal{V}_R(k) , \tag{2}$$

where

$$\mathcal{V}_R(k) = \mathcal{V}_0 f(k) . \tag{3}$$

In fact, according to the optical model, a nucleon of energy E and momentum K outside the nucleus, possesses inside a momentum

$$k(E) = K\left[1 - \frac{\mathcal{V}_R(E)}{E}\right]^{\frac{1}{2}}, \tag{4}$$

and the determination of the real part of the nuclear potential is in principle obtained by solving the equation

$$\mathcal{V}_R(E) = \mathcal{V}_0 f[k(E)] . \tag{5}$$

It is therefore clear that, if the unknown function $f(k)$ would be unity for any value of k, the real part of the potential would be independent of the nucleon energy E. Thus, the elastic scattering experiments of nucleons by nuclei suggest to abandon the statistical model based on Eq. (1). Since all states of lower energy are filled up in the nucleus, the internal momentum $k(E)$ of the incoming nucleon tends to $k(0)$ as the nucleon energy tends to zero. Therefore, the very problem in order to refine the statistical model of the nucleus is to extend the real part of the nuclear potential for $k \lesssim k_F = (2ME_F)^{\frac{1}{2}}$, preserving the continuity of $\mathcal{V}_R(k)$ at $k = k(0)$ with the zero energy extrapolation of the scattering data, which give information on the potential felt by a nucleon nearby the top of the Fermi sphere. Having neglected the

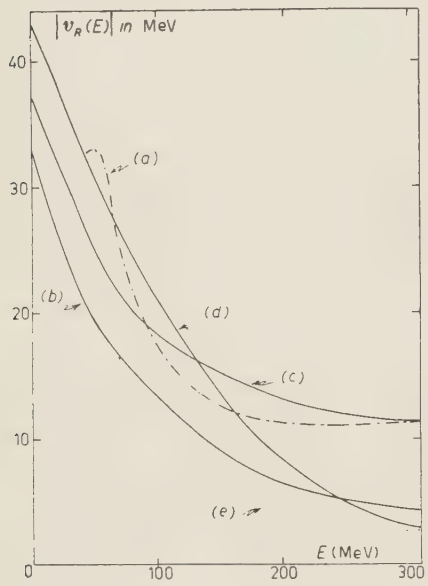

Fig. 1. – The real part of the nuclear potential $V_R(E)$ as a function of the neutron energy E. The curve (*a*) reproduces the energy dependence estimated by TAYLOR. The behaviour of $V_R(E)$, shown by curves (*b*), (*c*) and (*d*), has been obtained from Eqs. (14), (24) and (39) respectively. For comparison's sake, the behavior of $V_R(E)$, following from the Feshbach and Lomon phaseshifts (set A), has been also plotted (curve *e*) (see ref. [5]).

radial dependence of $\mathcal{V}_R(k)$, the forthcoming discussion on this problem will be confined to the limiting case of a large nucleus ([2]).

It is customary in the current use of the statistical model to evaluate $\mathcal{V}_0$ from Eq. (1), assuming $\mathcal{E}(k_F) = B$. Since, according to Eq. (2), one should expect that

$$B = \frac{k_F^2}{2M} + \mathcal{V}_R(k_F)\,, \tag{6}$$

it follows that in this kind of calculations the value of $\mathcal{V}_0$, used in Eq. (1), is correct in so far it is intended as the value of the real part of the nuclear potential at the top of the Fermi sphere. The correct choice of $\mathcal{V}_R(k)$ is therefore not only bound to be, through Eq. (5), in agreement with the elastic scattering data, but also with the saturation prescriptions of nuclear forces.

On calling $\langle T\rangle = (3/5)(k_F^2/2M)A$ the average value of the nucleus kinetic energy, and

$$\langle V\rangle = \tfrac{1}{2}\sum_{j=1}^{A}\mathcal{V}_R(k_j)\,, \tag{7}$$

the nucleus average potential energy, the average total energy $\langle W\rangle$ of the assembly of A nucleons is given by

$$\langle W\rangle = \frac{3A}{5}\left(\frac{k_F^2}{2M}\right) + \frac{3A}{2k_F^3}\int_0^{k_F}\mathcal{V}_R(k)k^2\,\mathrm{d}k\,. \tag{8a}$$

The necessary condition for $\langle W\rangle/A$ to be a minimum requires that $\mathcal{V}_R(k)$ satisfies the following equation

$$\frac{2k_F^2}{5M} + \mathcal{V}_R(k_F) = \frac{1}{3k_F^3}\int_0^{k_F}\left\{3\mathcal{V}_R(k) - k_F\frac{\partial\mathcal{V}_R(k)}{\partial k_F}\right\}k^2\,\mathrm{d}k\,. \tag{8b}$$

The minimum total energy per nucleon is thus given by

$$\frac{\langle W\rangle_{\min}}{A} = \frac{k_F^2}{2M} + \frac{1}{2}\left\{\mathcal{V}_R(k_F) + \frac{1}{k_F^2}\int_0^{k_F}k^2\frac{\partial\mathcal{V}_R(k)}{\partial k_F}\,\mathrm{d}k\right\}. \tag{8c}$$

If $\langle W\rangle_{\min}/A$ has to be equal to B — which is not so evident as usually believed — from Eqs. (6) and (8c) it follows that

$$k_F^2\mathcal{V}_R(k_F) = \int_0^{k_F}k^2\frac{\partial\mathcal{V}_R(k)}{\partial k_F}\,\mathrm{d}k\,. \tag{9}$$

([2]) W. S. Swiatecki: *Phys. Rev.*, **101**, 1321 (1956).

As will be shown below, we have been able to prove this remarkable relation only in particular cases.

In order to find out the way along which the conventional statistical model should be modified in agreement with the experimental evidence, we shall now investigate how the nucleon potential energy, constructed in terms of two-body forces, behaves in the interior of the nucleus.

Let us consider the nucleus as composed by $A/2$ protons and $A/2$ neutrons, uniformly distributed within a spherical box of volume $\Omega = (4\pi/3)R^3$. To simplify the discussion, the Coulomb energy and surface effects will be neglected, so that the two-body tensor interaction averages to zero and only the volume energy $B = -15$ MeV comes into play.

The real part of the nuclear potential, felt by a nucleon of momentum k within the nucleus, is given by

$$\mathcal{V}_R(k) = \int \Psi^* V \Psi \prod_{i=1}^{A} \mathrm{d}\boldsymbol{r}_i \Big/ \int \Psi^* \Psi \prod_{i=1}^{A} \mathrm{d}\boldsymbol{r}_i \,, \tag{10}$$

where $V(r)$ is the two-body potential and $\Psi(1, 2, \ldots A)$ is the ground state wave function for a nucleus of mass number A, expressed as a Slater determinant of free particle states, satisfying boundary conditions in the volume Ω. The individual nucleon wave functions are given by

$$\psi_j(m) = \Omega^{-\frac{1}{2}} \exp\left[i\boldsymbol{k}_m \boldsymbol{r}_j\right] \chi_j(\sigma_m) \nu_j(\tau_m) \,, \tag{11}$$

where $\chi_j(\sigma_m)$ and $\nu_j(\tau_m)$ are the well known two-row spinors associated to the m-th state of the j-th particle in the spin respectively isobaric spin space. The wave function of the nucleon having momentum k will be written without the suffix j. Eq. (10) becomes

$$\mathcal{V}_R(k) = \sum_{j=1}^{A} \int \mathrm{d}\boldsymbol{r}_1 \, \mathrm{d}\boldsymbol{r}_2 \psi^*(1) \psi_j(2) V(1, 2) \left[\psi(1)\psi_j(2) - \psi(2)\psi_j(1)\right] . \tag{12}$$

To carry out the calculation of Eq. (12) we need to explicit the two-nucleon potential. Unfortunately, among the very many potentials so far suggested, none of them is satisfactory. However, as will be shown later, for our purposes the choice of one potential rather than another is not found to be of importance. For this reason we choose the following central, static potential

$$V(r) = \frac{f^2}{3} (\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2) \frac{\exp\left[-\mu r\right]}{\mu r} \,. \tag{13}$$

which possesses many of the insufficiencies of more elaborated ones, as far as the saturation problem is concerned, but has the advantage to allow straight-

forward calculations. From Eqs. (12) and (13) readily [3] follows Eq. (3), where

$$\mathcal{V}_0 = -(3/\pi) f^2 k_F \mathcal{G}(0)\,, \qquad f(k) = \mathcal{G}(k)/\mathcal{G}(0)\,, \tag{14a}$$

$$\mathcal{G}(k) = 1 + \frac{\mu^2 + k_F^2 - k^2}{4kk_F} \ln \frac{\mu^2 + (k_F + k)^2}{\mu^2 + (k_F - k)^2} - \\ - \frac{\mu}{k_F} \operatorname{tg}^{-1}\left(\frac{k_F + k}{\mu}\right) - \frac{\mu}{k_F} \operatorname{tg}^{-1}\left(\frac{k_F - k}{\mu}\right), \tag{14b}$$

$$\mathcal{G}(0) = 2\{1 - (\mu/k_F) \operatorname{tg}^{-1}(k_F/\mu)\}\,. \tag{14c}$$

Taking into account that

$$\int_0^{k_F} k^2 \frac{\partial \mathcal{V}_R(k)}{\partial k_F}\, \mathrm{d}k = -\left(\frac{3f^2}{\pi}\right) k_F^3 \left\{1 + \frac{\mu^2}{4k_F^2} \ln \frac{\mu^2 + 4k_F^2}{\mu^2} - \frac{\mu}{k_F} \operatorname{tg}^{-1}\left(\frac{2k_F}{\mu}\right)\right\}, \tag{15}$$

Eq. (9) is proved to be satisfied, and from Eqs. (14) and (8*c*) one obtains Eq. (6). However, Eq. (8*b*), leading to $B = -15$ MeV, is not satisfied at the density minimizing $\langle W \rangle$. From Eq. (7) it is obtained [4]

$$\frac{\langle V \rangle}{A} = -\frac{9f^2 k_F}{4\pi} \left\{1 - \frac{\mu^2}{6k_F^2} + \frac{\mu^2}{2k_F^2}\left(1 + \frac{\mu^2}{12k_F^2}\right) \ln \frac{\mu^2 + 4k_F^2}{\mu^2} - \frac{4\mu}{3k_F} \operatorname{tg}^{-1}\left(\frac{2k_F}{\mu}\right)\right\}. \tag{16}$$

This function of the Fermi momentum k_F is in fact incapable to give the correct volume energy at the experimental nuclear radius, even if one abandons the preconceived idea that the saturation should be obtained with the same values of μ and f^2, which are required to fit the deuteron ground state (*). However, if we limit ourselves to get at normal density ($r_0 = \mu^{-1}$, $k_F = 1.52\mu$) the correct volume energy, regardless of the position of the minimum of the nucleus total energy, it is found $\mathcal{V}_R(k_F) = -40$ MeV. Since in this case $\mathcal{G}(0) = 0.72$, we have

$$\mathcal{V}_R(0) = \mathcal{V}_0 = -65.4 \text{ MeV}\,. \tag{17}$$

([3]) A. Kind and C. Villi: *Nuovo Cimento*, **1**, 749 (1955); H. A. Bethe: *Phys. Rev.*, **103**, 1353 (1956) (Sect. XI).

([4]) R. Huby: *Proc. Phys. Soc.*, A **62**, 62 (1949).

(*) Because of an oversight of one of the authors (C.V.) a factor (1/4) is missing in Eq. (5) of ref. ([3]). This correction does not alter the results concerning $\mathcal{V}_R(E)$, provided the strenght of the two-body potential is chosen in such a way to give the correct volume energy at normal nuclear density. The remark that Taylor's empirical curve is reproduced in agreement with Froelich's fit of the deuteron and triton ground state is therefore incorrect.

The real part of the nuclear potential, evaluated from Eq. (5) with this values of $\mathcal{V}_0$ and using the values of $f(k)$, listed in Table I, is in qualitative

TABLE I. - *Numerical values of the function* $f(k)$, *given in Eqs.* (14), *for* $k \geqslant k_F = 1.52\,\mu$.

k/μ	1.52	2.0	2.5	3.0	4.0	5.0	6.0
$f(k)$	0.612	0.440	0.313	0.226	0.133	0.087	0.061

agreement with Taylor's empirical determination (Fig. 1, curve *b*) and with the results obtained by other Authors [5]. The momentum dependence of $\mathcal{V}_R(k)$, given in Eqs. (14), can also be derived from a factorable potential. The result of BRUECKNER and WADA [6] for $\mathcal{V}_R(0)$ and $\mathcal{V}_R(k_F)$, given in their Eqs. (27), is obtained also from Eqs. (14) simply by decreasing the range of the two-body potential (13), *i.e.* by replacing μ with their parameter 2α. A different result is found if one introduces into Eq. (12) the S state Yamaguchi potential [7] $V(1,2) = \boldsymbol{O} v(1) v(2)$, where $\boldsymbol{O}$ is an exchange operator and $v(i)$ $(i=1,2)$ is a Yukawa function of range μ_0^{-1}. In this case, the momentum dependence of $\mathcal{V}_R(k)$ is given by Eq. (3), where

$$f(k) = \mathcal{G}(k)/\mathcal{G}(0)\,, \qquad \mathcal{G}(0) = 2\left[\operatorname{tg}^{-1}\left(\frac{k_F}{\mu_0}\right) - \frac{\mu_0 k_F}{\mu_0^2 + k_F^2}\right], \tag{18a}$$

$$\mathcal{G}(k) = \operatorname{tg}^{-1}\left(\frac{k_F + k}{\mu}\right) + \operatorname{tg}^{-1}\left(\frac{k_F - k}{\mu_0}\right) - \frac{\mu_0}{2k} \ln \frac{\mu_0^2 + (k_F + k)^2}{\mu_0^2 + (k_F - k)^2}\,. \tag{18b}$$

The saturation problem in terms of the potential (18) has been considered by SUNDERESAN [8].

In view of future discussions, it has to be noted that, as far as the determination of the correct volume energy is concerned, one may disregard the momentum dependence of the nucleon potential energy and replace Eq. (6) by

$$B = \frac{k_F^2}{2M^*} + \mathcal{V}_0\,, \tag{19}$$

where the fictitious mass M^* is a parameter to be determined. For instance the mass M^*, consistent with the potential (14), reads

$$\frac{M^*}{M} = \left\{1 + \frac{\mathcal{V}_0}{E_F \mathcal{G}(0)}\left[1 - \mathcal{G}(0) + \frac{\mu^2}{4k_F^2} \ln \frac{\mu^2 + 4k_F^2}{\mu^2} - \frac{\mu}{k_F} \operatorname{tg}^{-1}\left(\frac{2k_F}{\mu}\right)\right]\right\}^{-1}. \tag{20}$$

(5) W. B. RIESENFELD and K. M. WATSON: *Phys. Rev.*, **102**, 1157 (1956)
(6) K. A. BRUECKNER and W. WADA: *Phys. Rev.*, **103**, 1008 (1956).
(7) Y. YAMAGUCHI: *Phys. Rev.*, **95**, 1628 (1954).
(8) M. K. SUNDERESAN: *Phys. Rev.*, **105**, 1075 (1957).

At normal density ($E_F = 25$ MeV) Eq. (20) gives $M^*/M = 0.48$. This illustrative example ultimately suggests that if the potential depth is arbitrarily assumed constant within the nucleus (and equal to the depth at $k = 0$), the statistical model could still be based on Eq. (1), provided the nucleon mass M is replaced by the new mass M^*, *i.e.*

$$\mathcal{E}(k) = \frac{k^2}{2M^*} + \mathcal{V}_0 . \tag{21}$$

In order to give a meaning to the fictitious mass M^* and theoretical support to Eq. (21), it is important to point out a peculiar feature of the potential given in Eqs. (14). The function $f(k)$, evaluated for $0 \leqslant k \leqslant k_F = 1.52\mu$, practically behaves as $1 - \alpha k^2$, where $\alpha = 0.168/\mu^2$ (Table II). The dependence

TABLE II. – *Comparison between the values of $f(k)$ evaluated at normal nuclear density from Eqs.* (14) *and from the equation $1 - \alpha k^2$, where $\alpha = 0.168/\mu^2$.*

k/μ	0	0.3	0.6	0.9	1.2	1.52
Eqs. (14)	1	0.982	0.925	0.844	0.737	0.612
$1 - \alpha k^2$	1	0.985	0.939	0.864	0.758	0.612

of $f(k)$ on even powers of k only is, of course, not surprising, because it is required by the prescription of the nuclear potential to be invariant with respect to time reflection. What is important to stress is that a k^2-dependence is proved to be valid for *all* momenta inside the nucleus, and not only in the neighborhood of $k = 0$. Thus, assuming for reasons which will become clear later the new constant $r^* = (2\alpha)^{\frac{1}{2}}$, the potential (14) within the nucleus can be written as

$$\mathcal{V}_R(k) = \mathcal{V}_0(1 - \tfrac{1}{2}r^{*2}k^2) . \tag{22}$$

Introducing this potential into Eq. (2), one obtains Eq. (21) and the ratio M^*/M turns out to be

$$\gamma \equiv M^*/M = (1 - Mr^{*2}\mathcal{V}_0)^{-1} . \tag{23}$$

Before drawing any conclusion from the possibility to describe the nuclear potential, felt by a nucleon inside the nucleus, according to Eq. (22), it is necessary to establish to what extent this possibility is bound to the particular form of the potential (13). Several cases will now be examined with the aim to show that the peculiar features of the two-body potential have no importance at all as far as the momentum dependence of $\mathcal{V}_R(k)$ *inside*

the nucleus is concerned. This is the reason why in the forthcoming discussion we shall often use the old-fashioned two-body potential (13). As a limiting example let us prove that the behavior of $\mathcal{V}_R(k)$, given in Eq. (22), is not modified by the two-body, non-local but separable Yamaguchi potential. In fact, although Eqs. (18) are widely different from Eqs. (14), the function $f(k)$ can still be expressed as $1 - \alpha k^2$ for all momenta in the interior of the nucleus. Assuming [8] $\mu_0 = 2.040\mu$, in agreement with singlet and triplet low-energy data [7], it is found $\alpha = 0.172/\mu^2$ and the values obtained for $f(k)$ (Table III) are close to those listed in Table II.

TABLE III. – *Comparison between the values of $f(k)$ evaluated at normal nuclear density from Eqs.* (18) *and from the equation* $1 - \alpha k^2$ ($\alpha = 0.172/\mu^2$).

k/μ	0	0.3	0.6	0.9	1.2	1.52
Eqs. (18)	1	0.978	0.926	0.847	0.735	0.603
$1 - \alpha k^2$	1	0.985	0.938	0.861	0.753	0.603

Eq. (22) is characteristic for exchange forces. The real part of the potential, constructed, according to Eq. (12), from a pure Wigner force, cannot be written in the form (22), because the exchange density term tends to increase $\mathcal{V}_R(k)$ with increasing nucleon momentum k. It is easy to show that a mixture of ordinary and exchange forces leads to a $\mathcal{V}_R(k)$ which, inside the nucleus, behaves according to Eq. (22), provided special restrictions are imposed on the two-body potential strengths. This is immediately seen if, for instance, one assumes the ordinary and the exchange forces to be ruled in the coordinate space by a Yukawa function, because in this case the momentum dependence of the potential is given by Eqs. (14). In the particular case of a mixture of Wigner and Majorana forces, the condition that the latter one contribute four fifths and the Wigner force one fifth of the attraction in the deuteron ground state is sufficient for the validity of Eq. (22).

The particular choice of the two-body potential in Eq. (12) influences only the value of the parameter r^*, which, in the general case, is not independent of the potential strength. For example, adding the Wigner potential $V(r) = V_W \exp[-\mu r]/\mu r$ to the potential (13) ($\mu f^2/3 = -V_{\sigma\tau}$), from Eq. (12) it is found

$$\mathcal{V}_R(k) = (1/\pi)\{(2/3)(k_F/\mu)^3 V_W + (9V_{\sigma\tau} - V_W)(k_F/\mu)\mathcal{G}(0)f(k)\}, \tag{24}$$

where $f(k)$ and $\mathcal{G}(0)$ are given by Eqs. (14). In the interior of the nucleus this potential behaves according to Eq. (22), where

$$\mathcal{V}_0 = (1/\pi)\{(2/3)(k_F/\mu)^3 V_W + (9V_{\sigma\tau} - V_W)(k_F/\mu)\mathcal{G}(0)\}, \tag{25}$$

and the length r^* is transformed into the new parameter

$$\varrho^* = r^* \mathcal{G}^{\frac{1}{2}}(0)(9V_{\sigma\tau} - V_W)^{\frac{1}{2}}[(2/3)(k_F/\mu)^2 V_W + (9V_{\sigma\tau} - V_W)\mathcal{G}(0)]^{\frac{1}{2}} . \tag{26}$$

The high energy behavior of $\mathcal{V}_R(E)$, following from Eq. (24) and Eq. (5), is in satisfactory agreement with Taylor's empirical determination (Fig. 1, curve *c*), assuming at normal density

$$(9V_{\sigma\tau} - V_W)(k_F/\pi\mu)\mathcal{G}(0) = -65.4 \text{ MeV} , \qquad (2/3\pi)(k_F/\mu)^3 V_W = -7.2 \text{ MeV} , \tag{27}$$

i.e. $V_W = -9.7$ MeV and $V_{\sigma\tau} = -21.9$ MeV. It follows that $\varrho^* = 0.95 r^*$ and $\mathcal{V}_0 = -72.6$ MeV. Using Eq. (24) it is readily proved that Eq. (9), and therefore Eq. (6), is satisfied. As an obvious extension of the preceding considerations it may be remarked that the value of r^* is not independent of the meson-nucleon coupling constant if fourth order contributions to the two-nucleon potential are taken into account. For instance, using into Eq. (12) the Lévy-Klein two-body potential ([9]), $\mathcal{V}_R(k)$ is still found of the form (3), where

$$\mathcal{V}_0 = -(3/\pi)(G^2/4\pi)(\mu/2M)^2 k_F \mathcal{G}(0) , \qquad f(k) = \mathcal{G}(k)/\mathcal{G}(0) , \tag{28a}$$

$$\mathcal{G}(k) = 2k_F \int_{r_c}^{\infty} \mathrm{d}r \left\{ j_0(kr) j_1(k_F r) \exp[-\mu r] + \frac{3\pi a_1 (G^2/4\pi)}{2\mu k_F^2 r_0^3} \cdot \right. \\ \left. \cdot \left[1 - \frac{3 j_0(kr) j_1(k_F r)}{4 k_F r} \right] \left[\left(\frac{2}{\pi} \right) K_1(\mu r) - a_2 \left(\frac{\mu}{M} \right) \left(\frac{1}{\mu r} \right)^2 (1 + \mu r)^2 \exp[-2\mu r] \right\} . \tag{28b}$$

Assuming $a_1 = a_2$, the behavior of $\mathcal{V}_R(k)$ for $k \leqslant k_F$ is given by Eq. (22) with $r^* = 0.5/\mu$ ($r_c = 0.38/\mu$, $G^2/4\pi = 15$, $r_0 = \mu^{-1}$). The nucleus potential energy, evaluated by DRELL and HUANG ([10]) to discuss the capability of the Levy-Klein potential to saturate the nucleus, follows from Eqs. (7) and (28), assuming $a_2 = 0$. Eq. (22) has been found also capable of describing the momentum dependence of $\mathcal{V}_R(k)$, derived from the phenomenological potential used by CHRISTIAN, GAMMEL and THALER ([11]) to fit the nucleon-nucleon scattering data. In this case the mathematical expression of $\mathcal{V}_R(k)$ is simply given by a sum of relations like (28) with $a_1 = 0$, where the value of the two-body potential strength, range and core are properly chosen according to the parity and spin of the two-nucleon states.

The conclusion that Eq. (22) reproduces correctly the behavior of $\mathcal{V}_R(k)$ inside the nucleus is also supported by the examination of the momentum dependence of the real part of the potential implied by the phase shift de-

([9]) M. M. LÉVY: *Phys. Rev.*, **88**, 72 (1952); A. KLEIN: *Phys. Rev.*, **90**, 1101 (1953).
([10]) S. D. DRELL and K. HUANG: *Phys. Rev.*, **91**, 1527 (1953).
([11]) J. L. GAMMEL, R. S. CHRISTIAN and R. M. THALER: *Phys. Rev.*, **105**, 311 (1957).

scription of the nuclear interactions. In this case, assuming that the original scattering picture (12) can be corrected simply by replacing the tangents by the phase-shifts themselves (13), for $0 \leqslant k \leqslant k_F$ one has (14)

$$(29) \qquad \mathcal{V}_R(k) = -\frac{1}{\pi M}\left\{4\int_0^{\frac{1}{2}(k_F-k)} f_1(k')k'^2\,\mathrm{d}k' + \frac{1}{2k}\int_{\frac{1}{2}(k_F-k)}^{\frac{1}{2}(k_F+k)} f_1(k')[k_F^2-(2k'-k)^2]k'\,\mathrm{d}k'\right\},$$

where

$$(30) \qquad f_1(k) = f_{ss}(k) + 3f_{st}(k) + 3f_{ts}(k) + 9f_{tt}(k)\,,$$

being the four functions on the right-hand side of Eq. (30) the scattering amplitudes classified according to spin and isobaric spin substates. Denoting by K_L and δ_{JL} the phase-shifts for singlet respectively for triplet states, Eq. (30), up to the orbital angular momentum $L = 5$, reads

$$(31) \quad f_1(k) = k^{-1}\{3K_0 + 3K_1 + 15K_2 + 7K_3 + 24K_4 + 11K_5 + 3\delta_{01} + 3\delta_{10} + \\ + 3\delta_{12} + 9\delta_{11} + 15\delta_{21} + 15\delta_{23} + 20\delta_{22} + 7\delta_{32} + 7\delta_{34} + 21\delta_{33} + \\ + 27\delta_{43} + 36\delta_{44} + 15\delta_{54} + 33\delta_{55} + 39\delta_{65}\}\,.$$

Since the function $f_1(k)$, defined in Eq. (31), diverges at $k = 0$ because the deuteron bound state implies $\delta_{10} = \pi$ at $k = 0$, we shall consider the function $f_2(k) = kf_1(k)$. This function has been evaluated using the phase-shifts determined by Christian, Gammel and Thaler (11) (Fig. 2). The dependence of $\mathcal{V}_R(k)$ on even powers of k is readily proved by expanding the function $f_2(k)$ in power series of k, *i.e.*

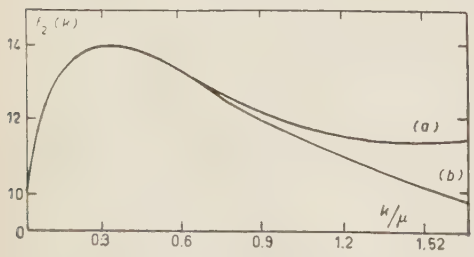

Fig. 2. – Behavior of $f_2(k) = kf_1(k)$ versus k in the interior of the nucleus. The curves (*a*) and (*b*) have been evaluated using all phase shifts respectively only S and P wave phase shifts determined by Christian, Gammel and Thaler (11).

$$(32) \qquad f_2(k) = \sum_{n=0} a_n k^n\,.$$

In fact, using Eq. (32) into Eq. (29) it is found

$$(33) \qquad \mathcal{V}_R(k) = -\frac{1}{\pi M}\sum_{n=0}\frac{k_F^{n+2}a_n}{(n+2)2^n}\cdot \\ \cdot\left\{1 + \frac{(n-1)(n+2)}{6k_F^2}k^2 + \sum_{i=0} c_{ij}k^{2j+4}\right\},$$

(12) K. A. Brueckner: *Phys. Rev.*, **96**, 508 (1954).
(13) B. S. DeWitt: *Phys. Rev.*, **103**, 1565 (1956).
(14) N. Fukuda and R. G. Newton: *Phys. Rev.*, **103**, 1558 (1956).

where

$$c_{nj} = \frac{n+2}{k_F^{2j+1}} \left\{ \binom{n+1}{2j+4} + \frac{1}{2}\binom{n}{2j+5} - \frac{1}{2}\binom{n+2}{2j+5} - \frac{1}{2}\binom{n}{2j+3} \right\}. \tag{34}$$

Numerical inspection of Eq. (29) shows that the influence of the k^{2j+4} terms, although increased by higher order waves, is negligible as compared to the k^2 term. It follows that Eq. (33) can be written in the form (22). The results of this calculation do not depend on a preconceived two-body potential, but nevertheless practically coincide with those following from the potential (13).

Before trying to give a logical content to Eq. (22), let us prove that this potential behavior is something more than a mere mathematical approximation of complicated functions of the momentum k, like Eqs. (14), (18), (24), (28) and (29), but it is indeed in satisfactory agreement with the elastic scattering xp eriments of low energy nucleons by nuclei. In fact, using Eq. (4) and Eq. (22) one has

$$\mathcal{V}_R(E) = \mathcal{V}_0[1 - \tfrac{1}{2} r^{*2} k^2(E)] , \tag{35}$$

and, finally, because of Eqs. (5) and (23), it is found

$$\mathcal{V}_R(E) = \frac{M^*}{M}\mathcal{V}_0 + \frac{M^*}{M}\left(\frac{M}{M^*} - 1\right) E . \tag{36}$$

Eq. (36), which may be considered as the optical model transformation of Eq. (22), is nothing but the nuclear potential given by Brueckner, Eden and Francis ([15]). Using the values ([15]) $\mathcal{V}_0 = -68$ MeV and $M^*/M = 0.6$, from Eq. (23) it is found $r^* = 0.45/\mu$, in close agreement with the value obtained from the potential (13). Since the zero energy limit $\mathcal{V}_R(0) = \gamma \mathcal{V}_0$ of Eq. (36) practically coincides with the potential $\mathcal{V}_R(k_F) = B - E_F$ at the top of the Fermi sphere, Eq. (9) suggests the following integral definition of the fictitious mass

$$\gamma = \frac{M^*}{M} \simeq \frac{1}{\mathcal{G}(0) k_F^2} \int_0^{k_F} k^2 \frac{\partial \mathcal{G}(k)}{\partial k_F} \, \mathrm{d}k . \tag{37}$$

For instance, taking into account Eq. (15), the two-body potential (13) gives

$$\frac{M^*}{M} \simeq \frac{1 + (\mu/2k_F)^2 \ln [1 + (2k_F/\mu)^2] - (\mu/k_F) \operatorname{tg}^{-1} (2k_F/\mu)}{2[1 - (\mu/k_F) \operatorname{tg}^{-1} (k_F/\mu)]} . \tag{38}$$

([15]) K. A. Brueckner, R. J. Eden and N. C. Francis: *Phys. Rev.*, **100**, 891 (1955).

In this approximation the ratio (38) is independent of the value of $\mathcal{V}_0$, which in Eq. (20) has been empirically chosen in order to have the correct volume energy at normal density.

As a final remark we point out that if Eq. (22) is arbitrarily regarded as the expansion for small momenta of the gaussian function $\mathcal{V}_R(k) = \mathcal{V}_0 \exp[-\frac{1}{2}r^{*2}k^2)$, one obtains FRAHN's approach ([16]). Such an expansion is correct provided $r^*k < \sqrt{2}$, *i.e.* in the interior of the nucleus and at normal density it must be $r^* < 0.9/\mu$. Using Eqs. (4) and (5), the gaussian extension of Eq. (22) gives

$$\mathcal{V}_R(E) = \mathcal{V}_0 \exp\left[\left(\frac{M}{M^*} - 1\right)\frac{E - \mathcal{V}_R(E)}{\mathcal{V}_0}\right]. \tag{39}$$

The behavior of $\mathcal{V}_R(E)$, given by Eq. (39), is shown in Fig. 1 (curve *d*) ($\mathcal{V}_0 = -68$ MeV and $M^*/M = 0.6$).

The importance we attach to the possibility of reproducing the behavior of $\mathcal{V}_R(E)$ in satisfactory agreement with nucleon-nucleus scattering experiments, using central, static two-body potentials, is mainly of a critical nature. Although the two-body forces are energy independent, the effective interactions between the particles in the nucleus depend on their state of motion through the exchange integral, involved by Eq. (12). It is indeed this exchange integral ([17]) which is responsible for the energy dependence of $\mathcal{V}_R(E)$ and the momentum dependence of $\mathcal{V}_R(k)$, given in Eq. (22). The choice of the two-body potential strongly affects the behavior of $\mathcal{V}_R(k)$ for $k > k_F$, but not for $k < k_F$, as clearly appears from the preceding calculations. The behavior of $\mathcal{V}_R(E)$ is in satisfactory agreement with the data, independently of the saturating nature of the two-body forces, provided the potential strength is chosen in such a way to give the correct volume energy at normal nuclear density. To resolve the internal contradiction of this situation one might be tempted to conclude that Eq. (5) is physically independent of Eqs. (8). Furthermore, Eq. (22), as it stands, does not satisfy Eq. (9) and cannot be used to check whether the saturation prescriptions of nuclear forces are fulfilled. The amazing result, which contributes to confuse rather than to clear up the situation, is that Eq. (22) — as will be seen in the next Section — can be logically justified without resorting to the invariance requirements of the nuclear potential, and suggests the clue for a remarkable improvement of the statistical model of the nucleus.

1·2. *The nucleon effective mass approximation.* – To understand the meaning of the empirical relation (22), let us summarize the preceding considerations. The basic assumption that the nucleons within the nucleus move freely in

([16]) W. E. FRAHN: *Nuovo Cimento*, **4**, 313 (1956).
([17]) J. M. VAN VLECK: *Phys. Rev.*, **48**, 367 (1935).

a constant potential well can still be retained even if one explicitely introduces the nuclear interactions, provided the expression of the nucleon kinetic energy is modified. Such a modification must take into account Eq. (3) in order to replace Eq. (1) by Eq. (21), where evidently in the general case the fictitious mass M^* is given by

$$\gamma = \frac{M^*}{M} = \left\{1 - \frac{2M\mathcal{V}_0}{k^2}[1 - f(k)]\right\}^{-1}. \tag{40}$$

Since $\mathcal{V}_0 < 0$ and $f(0) = 1$, the ratio γ is smaller than unity, and tends to the mass M of a free nucleon at the limit of large momenta. The definition (40) is clearly formal and has not an apparent physical meaning. The most logical way to shed light upon Eq. (40) is to take advantage of the fact that a strict analogy exists between the nuclear statistical model, based on Eq. (2), and the free electron model of metals [18].

Using wave packet methods, the relation between the group velocity $\boldsymbol{v}(\boldsymbol{k})$ of the packet and the energy $\mathcal{E}(\boldsymbol{k})$ of a nucleon is found to be

$$\boldsymbol{v}(\boldsymbol{k}) = \nabla_k \mathcal{E}(\boldsymbol{k}) . \tag{41}$$

Since the classical relation $\dot{\mathcal{E}} = \boldsymbol{F} \cdot \boldsymbol{v}$ between the applied force and the energy $\mathcal{E}$ should remain valid for the mean values of the quantum theory, one obtains

$$\langle \dot{\boldsymbol{v}} \rangle = \langle \nabla_k \boldsymbol{F} \cdot \boldsymbol{v} \rangle = \langle \boldsymbol{F} \cdot \nabla_k \nabla_k \mathcal{E}(\boldsymbol{k}) \rangle . \tag{42}$$

Eq. (42) is analogous to the classical equation $\langle \dot{\boldsymbol{v}} \rangle = \langle \boldsymbol{F} \rangle / M$ provided one introduces the following mass tensor

$$\frac{1}{\boldsymbol{M}^*} = \nabla_k \nabla_k \mathcal{E}(\boldsymbol{k}) . \tag{43}$$

If the nucleus is spherically symmetric and the nucleon energy depends only on the modulus k of the nucleon momentum, from Eq. (43) one obtains the following scalar quantity

$$M^* = \left\{\frac{\mathrm{d}^2 \mathcal{E}(k)}{\mathrm{d}k^2}\right\}^{-1}, \tag{44}$$

which will be called the *nucleon effective mass*. Using into Eq. (44) the total energy expressed by Eq. (2), it is found

$$\gamma = \frac{M^*}{M} = \left\{1 + M\mathcal{V}_0 \frac{\mathrm{d}^2 f(k)}{\mathrm{d}k^2}\right\}^{-1}, \tag{45}$$

---

[18] F. Seitz: *The Modern Theory of Solids* (New York, 1940).

from which it clearly appears that the nucleon effective mass M^* identifies with the nucleon mass M only when the nucleons are *free* ($\mathcal{V}_R = 0$) or when, according to Eq. (1), they move in a *constant* potential well ($\mathcal{V}_R = \mathcal{V}_0 = \text{const}$). The formal definition (40) is *mathematically equivalent* to the definition (44), following from the correspondence principle, provided $f(k)$ satisfies the following differential equation

$$k^2 \frac{\mathrm{d}^2 f(k)}{\mathrm{d}k^2} - 2f(k) = -2\,, \tag{46}$$

the solution of which, multiplied by $\mathcal{V}_0$, *i.e.* $\mathcal{V}_R(k)$, is just given by Eq. (22). Thus, if the potential felt by a nucleon in the nucleus behaves as an oscillator potential in momentum space, the modified nuclear particles, having total energy given by Eq. (2) and constant effective mass given by Eq. (23), play the role of nucleons in the independent particle model ([19]). It may be noted that from Eq. (23) one obtaines the following relation

$$r^{*2}|\mathcal{V}_0| = \frac{1}{M}\left(\frac{M}{M^*} - 1\right), \tag{47}$$

formally similar to that valid for the deuteron. Eqs. (21), (22) and (23) constitute what we shall call the *nucleon effective mass approximation* of the statistical model based on Eq. (2). According to this approximation, which is implicit in several recent works on heavy nuclei ([3,6,12,15,16,19,20]), Eq. (22) may be regarded as the expectation value of the potential energy operator $\mathcal{V}_0(1 + \frac{1}{2}r^{*2}\nabla^2)$: it follows that the total energy operator acts on the j-th nucleon wave function as

$$H_j = -\frac{\nabla_j^2}{2M^*} + \mathcal{V}_0\,. \tag{48}$$

The reason why in the preceding Section the parameter α has been replaced by $r^{*2}/2$ appears clear by solving Eq. (45) with γ constant ($f(0) = 1$ and $f(k) = 1$ at the limit $M^* = M$). We point out that the dependence of $\mathcal{V}_R(k)$ on k^4 does not follow from the correspondence principle and the equivalence criterion. Of course, even powers higher than k^2 may be introduced into Eq. (22) to improve the fit of $f(k)$, calculated from two-body forces [Eq. (12)] or on the basis of the phase-shift description of nuclear interactions [Eq. (29)], but their effect is negligible within the Fermi sphere. The influence of the k^4-term of the potential

$$\mathcal{V}_R(k) = \mathcal{V}_0 f(k) = \mathcal{V}_0(1 - \alpha k^2 + \beta k^4) \tag{49}$$

([19]) R. C. Eden and N. C. Francis: *Phys. Rev.*, **97**, 1366 (1955).

([20]) K. A. Brueckner: *Phys. Rev.*, **97**, 1353 (1955); W. E. Frahn: *Nuovo Cimento*, **5**, 393 (1957); W. E. Frahn and R. H. Lemmer: *Nuovo Cimento*, **5**, 523 (1957).

in reproducing the behavior of Eqs. (14) is shown in Table IV, which has to be compared with Table II. The parameters α and β are found to be

$$\alpha = 0.206/\mu^2, \qquad \beta = 0.016/\mu^4 . \tag{50}$$

TABLE IV. – *Comparison between the values $f(k)$ evaluated at normal nuclear density from Eqs.* (14) *and from equation* $1 - \alpha k^2 + \beta k^4$, *where* $\alpha = 0.206/\mu^2$ *and* $\beta = 0.016/\mu^4$.

k/μ	0	0.3	0.6	0.9	1.2	1.52
Eqs. (14)	1	0.982	0.925	0.844	0.737	0.612
$1 - \alpha k^2 + \beta k^4$	1	0.982	0.928	0.844	0.736	0.612

The real part of the nuclear potential at low energies is obtained performing the optical transformation of Eq. (49) by means of Eq. (4). It is found

$$\mathcal{V}_R(E) = E + \frac{1}{8M^*M\beta\mathcal{V}_0}\{1 - [1 + 16(E - \mathcal{V}_0)\mathcal{V}_0\beta M^{*2}]^{\frac{1}{2}}\}, \tag{51}$$

where $M^* = M(1 - 2M\alpha\mathcal{V}_0)^{-1}$ is the nucleon effective mass following from Eq. (44) in the case $\beta = 0$. Since $\beta < \alpha$, Eq. (51), up to terms in E^2, reads

$$\mathcal{V}_R(E) = \frac{M^*}{M}\mathcal{V}_0 + \left(1 - \frac{M^*}{M}\right)E + \frac{1}{2}\frac{M^{*3}\beta}{M\mathcal{V}_0}(E - \mathcal{V}_0)^2 , \tag{52}$$

which obviously identifies with Eq. (36) for $\beta = 0$. Eqs. (36) and (52), which involve nucleon momenta *higher* than k_F, are expected to be approximately correct for low energies only. The influence of the k^4-term on the nucleon potential energy will be not discussed in detail in this paper.

The substitution of Eq. (1) with Eq. (2), under the restrictions of the effective mass approximation, is clearly equivalent to add the term

$$\int \mu^{-2} f\varphi \sum_j |\nabla\psi_j|^2 \, \mathrm{d}\boldsymbol{r} , \tag{53}$$

to the nucleus Hamiltonian

$$H_1 = \int \{(2M)^{-1} \sum_j |\nabla\psi_j|^2 + \mu^2\varphi^2 - g\varphi \sum_j |\psi_j|^2\} \mathrm{d}\boldsymbol{r} . \tag{54}$$

Thus, the Johnson-Teller ([21]) definition of the nucleon effective mass

$$\frac{M^*}{M} = \left\{1 + \frac{2Mf\varphi}{\mu^2}\right\}^{-1} , \tag{55}$$

([21]) M. H. JOHNSON and E. TELLER: *Phys. Rev.*, **98**, 783 (1955).

is equivalent to the definition (23), following from Eq. (44). In fact, using the values of the quantities w and F, given in ref. ([21]), it is found $wF = r^{*2}|\mathcal{V}_0| = 3 \cdot 10^{-25}$ MeV·cm², *i.e.* $M^* = 0.5M$.

1·3. *Hydrodinamical evaluation of the nucleon effective mass.* – The correspondence principle, from which a sound theoretical definition of the nucleon effective mass has been derived [Eq. (43)], and the equivalence criterion [Eq. (46)], which has made possible to warrant the momentum dependence of $\mathcal{V}_R(k)$, given by Eq. (22), are both incapable of giving a hint concerning the value of the quantity M^*, except that it must be smaller than the free nucleon mass, if the potential $\mathcal{V}_0$ is attractive [Eq. (23)]. In Sect. 1·1 it has been shown how the effective mass ratio γ depends on the nucleus parameters through the function $\mathcal{G}(k)$ [Eq. (37)] and its value, calculated under very special assumptions [Eq. (20) and Eq. (38)], was found approximately included between 0.4 and 0.6, as several experimental data seem to require. The problem arises as to whether it is possible to give, on general arguments, an interpretation of the well established fact that the nucleon effective mass M^* appears to have approximately half the value of the free nucleon mass M. We shall now prove that the equation $M^* = 0.5M$ follows, as a limiting case, from a straightforward argument, based on classical hydrodynamics.

It is a matter of common experience that the apparent inertia of a body of mass M, moving in a fluid, is greatly increased by the fluid around it. This increased inertia is what is called the effective mass M^* of the body, and the difference between the effective mass and the real mass is usually called the « induced mass » M_i. According to classical hydrodynamics, the motion of a body in a fluid is dynamically equivalent to the motion of a *heavier* body in vacuo, whose effective mass is $M^* = M + M_i$. It is easily realized that this result, which was first given an exact mathematical formulation by Green and Stokes over a century ago ([22]), has to be changed if one attempts to describe, along the same lines, the motion of a nucleon in nuclear matter, because of the peculiar property of the nuclear fluid to exert attractive interactions upon the moving particle. Since in this case the increase of the kinetic energy of the fluid must be thought of as due to the nuclear interactions, an elementary energy balance convinces that the motion of a nucleon in nuclear matter is dynamically equivalent to the motion of a free nucleon having a *lighter* mass

$$M^* = M - M_i \,, \tag{56}$$

([22]) G. GREEN: *Mathematical Papers* (1833), p. 315; G. STOKES: *Mathematical and Philosophical Papers*, vol. I (1843), p. 17.

i.e. the effect of the attractive interactions between the nuclear body and the nuclear fluid is equivalent to a *negative* induced mass M_i, contrary to what has to be expected in connection with the motion of an ordinary body in an ordinary fluid. According to hydrodynamics, the induced mass M_i may be regarded as the mass of a virtual particle having a kinetic energy equal to the additional kinetic energy ΔT gained by the fluid. In order to evaluate ΔT, and therefore M_i, let us consider a nuclear particle of mass M and « radius » r_0, in motion with velocity v through a non-viscous and incompressible fluid of nuclear matter, extended over all space. Choosing the axis of spherical coordinates as the direction of the motion, the dipole velocity potential ([23]) reads

$$\mathcal{U} = -\frac{r_0^3 v \cos\vartheta}{2r^2} . \tag{57}$$

At any point of the nuclear fluid the radial and angular components of the velocity, following from the potential (57), are

$$v_r = \frac{r_0^3 v \cos\vartheta}{r^3} , \qquad v_\vartheta = \frac{r_0 v \sin\vartheta}{r^3} . \tag{58}$$

The additional kinetic energy of the nuclear matter is

$$\Delta T = \frac{1}{2}\int_{r \geq r_0} \varrho(\boldsymbol{r})(v_r^2 + v_\vartheta^2)\,\mathrm{d}\boldsymbol{r} . \tag{59}$$

Assuming a uniform distribution of nuclear matter ($\varrho = \varrho_0 = \text{const}$), Eq. (59) gives

$$\Delta T = \frac{1}{2}\left(\frac{2\pi\varrho_0 r_0^3}{3}\right) v^2 = \frac{1}{2} M_i v^2 . \tag{60}$$

Since the packing of A nucleons in the nucleus satisfies the law $R = r_0 A^{\frac{1}{3}}$, we have $\varrho_0 = 3M/4\pi r_0^3$. From Eq. (60) it is then found $M_i = M/2$ and finally Eq. (56) gives

$$M^* = 0.5M . \tag{61}$$

The translation of these considerations in quantum mechanics might probably be important for a further progress of the concept of the nucleon effective mass. This would require, as a preliminary step, to develop the calculation, which led us to the result stated in Eq. (61), without starting from the over-

---

([23]) G. Birkhoff: *Hydrodynamics* (Princeton, 1950).

simplified assumption of Dirichlet flow, in order to account for the effect due to the rotation of the body on the surrounding fluid.

1·4. *The radial dependence of the real part of the nuclear potential.* – The examination of the behavior in momentum space of the real part of the nuclear potential has shown (Sect. 1·1) that $\mathcal{V}_R(k)$ raises gradually from the full depth at $k = 0$ towards zero for very large nucleon momenta. The implicit assumption concerning its radial dependence was that it is a square well potential in co-ordinate space, *i.e.* $\mathcal{V}_R(k) = \mathcal{V}_0 f(k)$ for $r \leqslant R$ and $\mathcal{V}_R(k) = 0$ for $r > R$. Thus, the nuclear model based on Eq. (3) is a sharp-boundary model in the co-ordinate space. Such a model is not satisfactory for the interpretation of the differential cross-section of nucleons scattered by nuclei ([24]). On the other side, a nuclear potential $\mathcal{V}_R(r) = \mathcal{V}_0 \varphi(r)$, independent of k, is incompatible — as has been pointed out in Sect. 1·1 — with the energy dependence of the refractive index of nuclear matter for nucleon waves. The experimental data, therefore, suggest that $\mathcal{V}_R$ ought to depend both on the nucleon momentum k and on the radial distance r from the nucleus center, so that the real part of the potential has a full depth at $k = 0$ and $r = 0$ and dies away for large momenta and for large distances. Thus, a more realistic nuclear model should be a diffused boundary model both in momentum and co-ordinate space. It follows that Eq. (2) should be replaced by

$$\mathcal{E}(k, r) = k^2/2M + \mathcal{V}_R(k, r) , \tag{62}$$

where

$$\mathcal{V}_R(k, r) = \mathcal{V}_0 f(k, r) . \tag{63}$$

So far only empirical forms for $f(k, r)$ have been used, assuming that it can be splitted into the product of two functions, separately dependent on the momentum k and the radial distance r. According to this arbitrary assumption, Eq. (63) becomes

$$\mathcal{V}_R(k, r) = \mathcal{V}_0 f(k) \varphi(r) . \tag{64}$$

If one chooses

$$\varphi(r) = \left\{1 + \exp\left[\frac{r - R}{b}\right]\right\}^{-1} , \tag{65}$$

Eq. (64) reduces to the real part of the Saxon and Woods potential ([24]), successfully used in fitting the differential nucleon-nucleus scattering data. The

([24]) R. W. Woods and D. S. Saxon: *Phys. Rev.*, **95**, 577 (1954); **101**, 506 (1956); A. E. Glassgold, W. B. Cheston, M. L. Stein, S. B. Schould and G. W. Erikson: *Phys. Rev.*, **106**, 1207 (1957).

fit of the experiments requires the empirical determination of the parameter b and of $\mathcal{V}_R(k) = \mathcal{V}_0 f(k)$, as a function of the energy E of the scattered nucleons.

If one a priori believes that the radial and momentum separation, postulated in Eq. (64), is correct in the interior of the nucleus, $\mathcal{V}_R(k)$ can be replaced by Eq. (22), and one has

$$\mathcal{V}_R(k, r) = \mathcal{V}_0(1 - \tfrac{1}{2}r^{*2}k^2)\varphi(r) . \tag{66}$$

It follows that the nucleon total energy within the nucleus reads

$$\mathcal{E}(k, r) = \frac{k^2}{2M^*(r)} + \mathcal{V}_0\varphi(r) , \tag{67}$$

and the nucleon effective mass depends on the radial distance in the following way

$$\frac{M^*(r)}{M} = \left\{1 - Mr^{*2}\mathcal{V}_0\varphi(r)\right\}^{-1} . \tag{68}$$

Eq. (68) was first used by Ross, Lawson and Mark [25] and by Frahn and Lemmer [26].

Although in a more general approach the non-uniform character of the nuclear matter distribution should not be ignored, we shall here restrict ourselves to discuss the radial dependence of $\mathcal{V}_R(r)$ assuming a uniform distribution. Under this assumption, the calculation of the average kinetic energy of a nucleon in the nucleus allows the following definition of the average nucleon effective mass

$$\frac{\langle M^* \rangle}{M} = \left\{1 - 2Mr^{*2}\frac{\langle V \rangle}{A}\right\}^{-1} , \tag{69}$$

which is similar to Eq. (23), except for the substitution of $\mathcal{V}_0$ with $2\langle V \rangle / A$.

1·5. *The nucleon radial effective mass approximation.* – We shall now evaluate the nuclear potential $\mathcal{V}_R(k, r)$ in terms of two-body forces and on the basis of the same assumptions adopted in Sect. 1·1 to evaluate $\mathcal{V}_R(k)$. In this case the real part of the potential is made up by an ordinary and an exchange contribution

$$\mathcal{V}_R(k, r) = \mathcal{V}_R^{(\mathrm{ord})}(r) + \mathcal{V}_R^{(\mathrm{exch})}(k, r) . \tag{70}$$

---

[25] A. A. Ross, R. D. Lawson and H. Mark: *Phys. Rev.*, **104**, 401 (1957).
[26] W. E. Frahn and R. H. Lemmer: *Nuovo Cimento*, **5**, 1564 (1957).

Be $V(r'') = \boldsymbol{O}v(r'')$ the two-body potential, where $\boldsymbol{O}$, measured in MeV, is an exchange operator, $v(r'')$ the radial function and $r'' = |\boldsymbol{r} - \boldsymbol{r}'|$ the internucleon distance. The ordinary and the exchange contributions to the real part of the potential are found to be

$$\mathcal{V}_R^{(\mathrm{ord})}(r) = (A/\Omega) \sum_{\sigma\tau}^{(\mathrm{ord})} \int v(r'')\,\mathrm{d}\boldsymbol{r}', \tag{71a}$$

$$\mathcal{V}_R^{(\mathrm{exch})}(k,r) = (2\pi)^{-3} \sum_{\sigma\tau}^{(\mathrm{exch})} \int \exp\left[-i(\boldsymbol{k} - \boldsymbol{k}')\boldsymbol{r}''\right] v(r'')\,\mathrm{d}\boldsymbol{r}'\,\mathrm{d}\boldsymbol{k}', \tag{71b}$$

where the quantities $\sum_{\sigma\tau}^{(\mathrm{ord})}$ and $\sum_{\sigma\tau}^{(\mathrm{exch})}$ are obtained by summing up over spin and isobaric spin states. To evaluate Eqs. (71) it is convenient to invert in the co-ordinate space the order of integrations over $\boldsymbol{r}'$ and $\boldsymbol{r}''$ ($\mathrm{d}\boldsymbol{r}' = (2\pi/r)r'\,\mathrm{d}r'\,r''\,\mathrm{d}r''$). This operations has to be done according to the following rules

$$\int_0^R \mathrm{d}r' \int_{r-r'}^{r+r'} \mathrm{d}r'' \to \int_{r-R}^{r} \mathrm{d}r'' \int_{r-r''}^{R} \mathrm{d}r' + \int_{r}^{r+R} \mathrm{d}r'' \int_{r''-r}^{R} \mathrm{d}r' . \qquad (r \geqslant R); \tag{72}$$

If $r' \leqslant r$, one has

$$\int_0^r \mathrm{d}r' \int_{r-r'}^{r+r'} \mathrm{d}r'' \to \int_0^r \mathrm{d}r'' \int_{r-r''}^{r} \mathrm{d}r' + \int_r^{2r} \mathrm{d}r'' \int_{r''-r}^{r} \mathrm{d}r' . \qquad (r \leqslant R). \tag{73a}$$

If $r' \geqslant r$, one has

$$\int_0^R \mathrm{d}r' \int_{r'-r}^{r'+r} \mathrm{d}r'' \to \int_0^{R-r} \mathrm{d}r'' \int_{r}^{r''+r} \mathrm{d}r' + \int_{R-r}^{2r} \mathrm{d}r'' \int_{r}^{R} \mathrm{d}r' + \int_{2r}^{R+r} \mathrm{d}r'' \int_{r''-r}^{R} \mathrm{d}r' . \qquad (r \leqslant R). \tag{73b}$$

Taking into account Eqs. (72) and (73) and performing the integration over $\boldsymbol{k}'$, Eq. (71a) becomes

$$\mathcal{V}_R^{(\mathrm{ord})}(r) = \mathcal{V}_0^{(\mathrm{ord})} \varphi_{\mathrm{ord}}(r) , \tag{74}$$

where all constant factors have been lumped into $\mathcal{V}_0^{(\mathrm{ord})}$, while

$$\varphi_{\mathrm{ord}}(r) = \frac{1}{r} \int_{r-R}^{r+R} (R^2 - r^2 + 2rr'' - r''^2) v(r'')\,\mathrm{d}r'', \qquad (r \geqslant R), \tag{75a}$$

$$\varphi_{\mathrm{ord}}(r) = \frac{1}{r} \left\{ \int_0^{R-r} r''(2r + r'')v(r'')\,\mathrm{d}r'' + \right.$$

$$\left. + \int_0^{R+r} r''(2r - r'')v(r'')\,\mathrm{d}r'' + (R^2 - r^2) \int_{R-r}^{R+r} v(r'')\,\mathrm{d}r'' \right\}. \qquad (r \leqslant R). \tag{75b}$$

Eq. (71*b*) becomes

$$\mathcal{V}_R^{(\mathrm{exch})}(k, r) = \mathcal{V}_0^{(\mathrm{exch})} \varphi_{\mathrm{exch}}(k, r) , \tag{76}$$

where

$$\varphi_{\mathrm{exch}}(k, r) = \frac{1}{r} \int_{r-R}^{r+R} j_0(kr'') j_1(k_F r'')(R^2 - r^2 + 2rr'' - r''^2) v(r'') \frac{\mathrm{d}r''}{r''} , \qquad (r \geqslant R) , \tag{77a}$$

$$\varphi_{\mathrm{exch}}(k, r) = \frac{1}{r} \left\{ \int_0^{R-r} j_0(kr'') j_1(k_F r'')(2r + r'') v(r'') \, \mathrm{d}r'' + \right.$$

$$\left. + \int_0^{R+r} j_0(kr'') j_1(k_F r'')(2r - r'') v(r'') \, \mathrm{d}r'' + (R^2 - r^2) \int_{R-r}^{R+r} j_0(kr'') j_1(k_F r'') v(r'') \frac{\mathrm{d}r''}{r''} \right\} , \qquad (r \leqslant R) , \tag{77b}$$

being $j_n(x)$ spherical Bessel functions [27]. On calling

$$\mathcal{V}_0 = 4 \int_0^R \left\{ \mathcal{V}_0^{(\mathrm{ord})} r'' + j_1(k_F r'') \mathcal{V}_0^{(\mathrm{exch})} \right\} v(r'') \, \mathrm{d}r'' , \tag{78}$$

the results of the preceding calculation can be written in the compact form (63), where

$$f(k, r) = \frac{\mathcal{V}_0^{(\mathrm{ord})} \varphi_{\mathrm{ord}}(r) + \mathcal{V}_0^{(\mathrm{exch})} \varphi_{\mathrm{exch}}(k, r)}{\mathcal{V}_0^{(\mathrm{ord})} \varphi_{\mathrm{ord}}(0) + \mathcal{V}_0^{(\mathrm{exch})} \varphi_{\mathrm{exch}}(0, 0)} . \tag{79}$$

Inspection of Eqs. (77) shows that, for a given value of r, $\varphi_{\mathrm{exch}}(k, r)$ is a decreasing function of the nucleon momentum k, while, for a given value of k, $\varphi_{\mathrm{exch}}(k, r)$ decreases with increasing r. The possibility of disentangling, according to Eq. (64), the dependence of $\varphi_{\mathrm{exch}}(k, r)$ on k and r so that $\varphi_{\mathrm{exch}}(k, r) = f(k)\varphi(r)$ does not appear as possible. Since in the interior of the nucleus the internucleon distance varies from 0 to $2R$, the function $j_0(kr'')$ in Eq. (77*b*) can be expanded in series for nucleon momenta satisfying the condition $2kR < 1$. For a heavy nucleus this inequality holds for k smaller than about 1/10 of the Fermi momentum k_F. Thus, writing $j_0(kr'') = 1 - (kr'')^2/3 + \ldots$, Eq. (77*b*) reveals, in agreement with the considerations of Sect. 1·1 and Sect. 1·2, the dependence of the exchange contribution on even powers, practically up to the second, of the nucleon momentum k. A careful examination of Eq. (77*b*) shows that this is also true for nucleon momenta as large as k_F, although it cannot be put in explicit mathematical form. In fact, Eq. (77*b*) can be written as

$$\varphi_{\mathrm{exch}}(k, r) = \varphi_{\mathrm{exch}}(0, 0)[1 - \tfrac{1}{2} l^2(r) k^2] , \tag{80}$$

---

[27] L. I. Schiff: *Quantum Mechanics* (New York, 1949), p. 77.

but the function $l(r)$ must be determined numerically by inspection of the double set of curves $k = \text{const}$ and $r = \text{const}$. Thus, if one neglects the ordinary contribution to $\mathcal{V}_R(k, r)$ and takes into account Eq. (79), Eq. (63) becomes

$$\mathcal{V}_R(k, r) = \mathcal{V}_0[1 - \tfrac{1}{2}l^2(r)k^2] . \tag{81}$$

The nucleon total energy and the nucleon effective mass, following from Eq. (81), are

$$\mathcal{E}(k, r) = \frac{k^2}{2M^*(r)} + \mathcal{V}_0 , \tag{82}$$

$$\gamma(r) = \frac{M^*(r)}{M} = \{1 - M\mathcal{V}_0 l^2(r)\}^{-1} . \tag{83}$$

Eq. (83) is fundamentally different from Eq. (68), because the function $l(r)$ is bound to the radial dependence of the nuclear potential in a far more complicated way than the function $q(r)$. Eqs. (81), (82) and (83) constitute what we shall call *the nucleon radial effective mass approximation*. The relation between this approximation and that discussed in Sect. 1·2 can be readily established by calculating the average kinetic energy per nucleon. In this way it is possible to define the following quantity as the average nucleon effective mass

$$\frac{\langle M^* \rangle}{M} = \{1 - M\mathcal{V}_0\langle l^2 \rangle\}^{-1} , \tag{84}$$

where

$$R^3\langle l^2 \rangle = 3\int_0^R r^2 l^2(r)\,\mathrm{d}r . \tag{85}$$

It follows that Eq. (84) reduces to Eq. (23) by identifying the average value of the function $l^2(r)$ with the parameter r^{*2}.

1·6. *The imaginary part of the nuclear potential.* – It is known that the complex square well description of the nucleus ([28]) is greatly improved by a rounded edge potential, which eliminates certain discrepancies between the calculated and the experimental angular distributions of the scattered nucleons, by decreasing the reflection of the incoming wave on the nuclear surface. The diffused surface optical model is usually based on the complex potential

$$\mathcal{V}(E, r) = [\mathcal{V}_R(E) + i\mathcal{V}_I(E)]\varphi(r) = \mathcal{V}_R(E)[1 + i\zeta(E)]\varphi(r) , \tag{86}$$

([28]) H. Feshbach, C. E. Porter and V. E. Weisskopf: *Phys. Rev.*, **96**, 448 1954).

where $\zeta(E)$ is the nucleon absorption parameter defined as the ratio $\mathcal{V}_I(E)/\mathcal{V}_R(E)$ of the imaginary and real part of the potential. The function $\varphi(r)$, dependent on the nuclear radius and on the diffuseness length, is usually assumed of the form given in Eq. (65).

The physical information one can derive from the fit of the data using Eq. (86) are rather provisional, being no theoretical justification for the separation of the energy dependent from the radial part of the complex potential $\mathcal{V}(E, r)$. On the contrary, even the simplest possible calculation, like that developed in Sect. 1·5, shows that such a separation is certainly not correct for the real part of the potential. This can be easily recognized by noting that $\mathcal{V}_R(E, r)$ is obtained from Eq. (70) ($k \geqslant k_F$) by replacing k with $k(E)$ and then solving Eq. (5). Since the calculation of the imaginary part of the potential cannot be carried out ignoring $\mathcal{V}_R(E, r)$, one should expect that such a separation is also objectionable for $\mathcal{V}_R(E, r)$.

The evaluation of $\mathcal{V}_I(E, r)$, taking into account the radial and energy dependence of the real part of the potential, is an extremely difficult problem, which is beyond the scope of the present investigation. We shall here restrict ourselves only to refine the calculation of $\mathcal{V}_I(E)$, carried out in a preceding paper [29]. Taking into account only the energy dependence of $\mathcal{V}_R(E)$, the hypothesis of free particle interactions, underlying the calculation of ref. [29], does not alter the final result, except for the fact that the momentum $k(E)$ of the incident nucleon inside the nucleus has to be expressed by means of Eq. (4), all mathematical restrictions due to the energy conservation and the Pauli principle being unchanged. Assuming for the energy dependence of the experimental neutron-proton total cross-section an expression of the type $\sigma(E) = c_0(c_1+E)^{-1}$, where $c_0 = 8.64$ MeV·barns and $c_1 = 1.08$ MeV, the final result reads

$$\mathcal{V}_I(E) = (5Mc_0/16\pi^2)vk(E)F(E)\,, \tag{87}$$

v being the velocity of the incident nucleon and

$$F(E) = \int^{x_F} \frac{x(1 - x^2 + 2x_F)}{[2(1 + x^2 + y)]^{\frac{1}{2}}} \ln \frac{1 + 3x^2 + 2y + 2x[2(1 + x^2 + y)]^{\frac{1}{2}}}{1 + 3x^2 + 2y - 2x[2(1 + x^2 + y)]^{\frac{1}{2}}}\, \mathrm{d}x\,, \tag{88}$$

where $x = k/k(E)$, $x_F = k_F/k(E)$ and $y = Mc_1/k^2(E)$. The lower limit of integration in Eq. (88) is $(2x_F^2 - 1)^{\frac{1}{2}}$ for $E - \mathcal{V}_R(E) \leqslant E_F$, otherwise zero.

For low energies, $\mathcal{V}_R(E)$ can be expressed through Eq. (36), and therefore one has $k^2(E) = 2M^*(E - \mathcal{V}_0)$. It follows that in this approximation the imaginary part of the potential depends separately on M^* and $\mathcal{V}_0$, and con-

---

[29] E. Clementel and C. Villi: *Nuovo Cimento*, **2**, 176 (1955). For a different approach to the problem see M. Cini and S. Fubini: *Nuovo Cimento*, **2**, 75 (1955).

sequently from the low energy scattering data it should be possible to obtain information on these two quantities, which could provide an experimental value for the parameter r^*, defined in Eq. (47).

Although the model adopted for the evaluation of $\mathcal{V}_I(E)$ is crude, it is rather gratifying that a better agreement with the data is obtained with these « frivolous » calculations ([29,30]) than with « serious » ones.

2. – The modified statistical model of the nucleus.

2·1. *The nuclear excitation energy.* – The energy dependence of the real part of the nuclear potential inside the nucleus implies important modifications on the nuclear excitation energy. The distribution of nucleon momenta at the nuclear temperature τ (in MeV) is given by $g(k, \tau)\,\mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{r}/(2\pi)^3$, where $g(k, \tau)$ is the well known Fermi function. For generality we assume the nucleon total energy and the nucleon effective mass to be expressed by Eq. (2) and (40) respectively. In order to avoid mathematical complications, we shall regard the parameter M^* as dependent on the nucleon kinetic energy rather than on the nucleon momentum k, *i.e.* $\gamma(\varepsilon) = M^*(\varepsilon)/M$.

Introducing a new parameter γ_0 related to γ, defined in Eq. (40), by the following equation, which accounts for the factor $\frac{1}{2}$ appearing in Eq. (7),

$$\gamma_0 = \frac{2\gamma}{1+\gamma}, \tag{89}$$

the well known ([31]) distribution of energy levels for the individual particles becomes

$$n(\varepsilon)\mathrm{d}\varepsilon = (\tfrac{3}{2})C\Phi(\varepsilon, \mathcal{V}_0)g(\varepsilon, \tau)\varepsilon^{\frac{1}{2}}\,\mathrm{d}\varepsilon\,, \tag{90}$$

where $C = (2^{\frac{7}{2}}/9\pi)(MR^2)^{\frac{3}{2}}$ and

$$\Phi(\varepsilon, \mathcal{V}_0) = \gamma_0^{\frac{3}{2}}(\varepsilon)\left\{1 + \varepsilon\,\frac{\mathrm{d}\gamma_0(\varepsilon)}{\mathrm{d}\varepsilon}\right\}, \tag{91a}$$

$$g(\varepsilon, \tau) = \left\{\exp\left[(\varepsilon - \varepsilon_F^*)/\tau\right] + 1\right\}^{-1}, \tag{91b}$$

ε_F^* being the new Fermi energy at the temperature τ. To simplify the notation, we shall indicate with the index $i = 1, 2$ the quantities related to protons respectively neutrons ($N_1 \equiv Z$, $N_2 \equiv N$).

Following standard statistical methods, the number N_i and the energy

([30]) A. M. Lane and F. Wandel: *Phys. Rev.*, **98**, 1524 (1955).
([31]) H. A. Bethe: *Rev. Mod. Phys.*, **9**, 69 (1937).

$W^*(N_i, \tau)$ of each of the two strongly degenerated nucleon assemblies is found to be

$$N_i = C\varepsilon_{F,i}^{*\frac{3}{2}}\{\lambda_1(\mathcal{V}_{0i}) + (\pi^2/8)\nu_1(\mathcal{V}_{0i})(\tau/\varepsilon_{F,i}^*)^2\}, \tag{92}$$

$$W^*(N_i, \tau) = (3/5)C\varepsilon_{F,i}^{*\frac{5}{2}}\{\lambda_2(\mathcal{V}_{0i}) + (5\pi^2/8)\nu_2(\mathcal{V}_{0i})(\tau/\varepsilon_{F,i}^*)^2\}. \tag{93}$$

The functions $\lambda_n(\mathcal{V}_{0i})$ and $\nu_n(\mathcal{V}_{0i})$ are defined as

$$\lambda_n(\mathcal{V}_{0i}) = \frac{2n+1}{2}\int_0^1 x^{(2n+1)/2}\Phi(x, \mathcal{V}_{0i})\,\mathrm{d}x\,, \tag{94}$$

$$\nu_n(\mathcal{V}_{0i}) = \frac{2}{2n-1}\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{(2n-1)/2}\,\Phi(x, \mathcal{V}_{0i})\right]\right\}_{x=1}, \tag{95}$$

being $x = \varepsilon/\varepsilon_{F,i}^*$.

For $\tau = 0$ Eq. (92) defines the Fermi energy for both protons and neutrons in the nucleus ground state

$$E_{F,i}^* = \lambda_1^{-\frac{2}{3}}(\mathcal{V}_{0i})(N_i/C)^{\frac{2}{3}} = \lambda_1^{-\frac{2}{3}}(\mathcal{V}_{0i})(2N_i/A)^{\frac{2}{3}}E_F\,, \tag{96}$$

where $E_F = (A/2C)^{\frac{2}{3}}$ is the Fermi energy for a standard nucleus ($N_i = A/2$, $\gamma_0 = 1$). Since the normalization of the Fermi distribution at the temperature τ requires, for low excitation energies, that the parameter $\varepsilon_{F,i}^*$ be related to the Fermi energy at $\tau = 0$ by the following equation

$$\varepsilon_{F,i}^* = E_{F,i}^*\{1 - (\pi^2/12)[\nu_1(\mathcal{V}_{0i})/\lambda_1(\mathcal{V}_{0i})](\tau/E_{F,i}^*)^2\}, \tag{97}$$

Eq. (93), up to terms in τ^2, becomes

$$W^*(N_i, \tau) = (3/5)CE_{F,i}^{*\frac{5}{2}}[\lambda_2(\mathcal{V}_{0i}) + (5\pi^2/12)\nu_2(\mathcal{V}_{0i})(\tau/E_{F,i}^*)^2]\,. \tag{98}$$

The zero point energy of the nucleus is

$$W_0^* = (3A/10)E_F\sum_{i\ 1.2}\lambda_1^{-\frac{5}{3}}(\mathcal{V}_{0i})\lambda_2(\mathcal{V}_{0i})(2N_i/A)^{\frac{5}{3}}\,, \tag{99}$$

and therefore the excitation energy

$$Q(\tau) = \sum_{i=1.2} W^*(N_i, \tau) - W_0^*\,, \tag{100}$$

reads

$$Q(\tau) = \{(\pi^2 A/8E_F)\sum_{i=1.2}\lambda_1^{-\frac{1}{3}}(\mathcal{V}_{0i})\nu_2(\mathcal{V}_{0i})(2N_i/A)^{\frac{1}{3}}\}\tau^2\,. \tag{101}$$

Before going further, let us consider the potential (49), which according to Eq. (40), implies a nucleon effective mass dependent on the momentum k. On the basis of the preceding considerations it is clear that the following quantity has to be considered as the nucleon « kinetic » energy

$$\varepsilon = \frac{k^2}{2M}(1 - M\alpha\mathcal{V}_0 + M\beta\mathcal{V}_0 k^2), \tag{102}$$

from which the momentum k and, therefore, the nucleon effective mass, can be expressed as a function of the energy. It is found

$$\gamma_0(\varepsilon) = 2\{(1 - M\alpha\mathcal{V}_0) + [(1 - M\alpha\mathcal{V}_0)^2 + 8M^2\beta\mathcal{V}_0\varepsilon]^{\frac{1}{2}}\}^{-1}. \tag{103}$$

Inspection of Table V shows that $\gamma_0(\varepsilon)$ and the function $\Phi(\varepsilon, \mathcal{V}_0)$, defined in Eq. (91*a*), vary slowly with the energy in the interval $0 \leqslant \varepsilon \leqslant 50.7$ MeV, where 50.7 MeV is the new Fermi energy E_F^*, corresponding to the potential (49).

TABLE V. – *Energy dependence of $\gamma_0(\varepsilon)$, $\gamma(\varepsilon)$ and $\Phi(\varepsilon, \mathcal{V}_0)$ in the interior of the nucleus. The values of $\mathcal{V}_0$, α, β are given in Eqs.* (17) *and* (50) *respectively.*

ε (MeV)	0	10	20	30	40	50.7
$\gamma_0(\varepsilon)$	0.601	0.613	0.625	0.638	0.651	0.669
$M^*(\varepsilon)/M$	0.429	0.442	0.454	0.468	0.482	0.503
$\Phi(\varepsilon, \mathcal{V}_0)$	0.465	0.484	0.503	0.532	0.556	0.595

Although the small variation of $\gamma_0(\varepsilon)$ and $\Phi(\varepsilon, \mathcal{V}_0)$ with the energy is not entirely negligible, we shall simplify the theory by neglecting the k^4-term in the nuclear potential. The possibility to apply the nucleon effective mass approximation to an excited nucleus is by no means obvious. In fact, the existence of high momentum components in the nucleon momentum distribution for $\tau \neq 0$ extends the variation of k from 0 to ∞ and, therefore, the nucleon effective mass approximation, which is valid only for $0 \leqslant k \leqslant k_F$, is apparently inapplicable. However, the mathematical reason why this approximation can be safely used for an excited nucleus is readily found by noting that the Fermi approximation ([32]), used in deriving Eq. (101), restricts the variation of the nucleon energy from zero up to $\varepsilon^*_{F,i}(\tau)$, given by Eq. (97), *i.e.* in an energy interval where Eq. (22) is certainly valid, the maximum Fermi momentum $k_{F,i}(\tau) = [2M^*\varepsilon^*_{F,i}(\tau)]^{\frac{1}{2}}$ being smaller than the Fermi momentum at $\tau = 0$. Then, assuming $\gamma_0(\varepsilon) = \text{const}$, one has $\lambda_n(\mathcal{V}_{0i}) = \nu_n(\mathcal{V}_{0i}) = \gamma_0^{\frac{3}{2}}$ and

([32]) J. E. MAYER and M. G. MAYER: *Statistical Mechanics* (New York, 1940).

Eqs. (96), (99) and (101) become

$$E^*_{F,i} = \gamma_0^{-1}(2N_i/A)^{\frac{2}{3}}E_F\,, \tag{104}$$

$$W^*_0 = (3A/10)E_F \sum_{i=1,2} \gamma_0^{-1}(2N_i/A)^{\frac{5}{3}}\,, \tag{105}$$

$$Q(\tau) = a^*\tau^2\,, \tag{106}$$

where

$$a^* = (\pi^2/8E_F)[(2Z/A)^{\frac{2}{3}} + (2N/A)^{\frac{2}{3}}]\gamma_0 A\,. \tag{107}$$

The nuclear excitation energy for a standard nucleus becomes simply

$$Q(\tau) = (\pi^2 A/4E_F)\gamma_0\tau^2 = a\gamma_0\tau^2\,, \tag{108}$$

where a is the level density parameter of the conventional statistical model.

The calculation of the excitation energy on the basis of the nucleon radial effective mass approximation, discussed in Sec. 1·5, is straightforward. One of the most remarkable consequences of the radial dependence of the nucleon effective mass is that the maximum Fermi energy becomes dependent on the distance r from the nucleus center, while the maximum Fermi momentum k_F remains unchanged and retains the value given by the conventional model. Introducing these peculiar features into the Fermi approximation, the nuclear excitation energy is found to be

$$Q(\tau) = \left\{\frac{3a}{R^3}\int_0^R \gamma_0(r)r^2\,\mathrm{d}r\right\}\tau^2 = a\langle\gamma_0\rangle\tau^2\,, \tag{109}$$

where $\langle\gamma_0\rangle$ is the average value of $\gamma_0(r)$ obtained replacing Eq. (83) into Eq. (89). Eq. (109) obviously identifies with Eq. (108) if the nucleon effective mass is assumed independent of r. Since $\gamma_0(r)$ depends on the function $l(r)$, Eq. (109) establishes a relationship between the nuclear level density and the radial dependence of the nuclear potential inside the nucleus.

Assuming $M^*/M = 0.5$ ($\gamma_0 = 0.667$), Eq. (107) gives, for a standard nucleus of radius $R = 1.22\cdot 10^{-13}A^{\frac{1}{3}}$ cm,

$$a^* = 0.050A\ (\mathrm{MeV})^{-1}\,. \tag{110}$$

This value of a^* coincides with that empirically determined by FONG ([33]), thus showing that his statistical theory of fission implicitely underlies the

([33]) P. FONG: *Phys. Rev.*, **102**, 434 (1956).

nucleon effective mass approximation. In this connection it may be noted that the value $\gamma \sim 0.5$ ($\gamma_0 \sim 0.667$) is also required to remove the discrepancy by a factor 2 between the calculated ([34]) and the observed principal resonances for γ-ray absorption, and to account for the rotational levels of nuclei ([35]).

The important role of the parameter γ_0 in fitting several experimental data is better understood by examining Eq. (108). Its physical meaning becomes clearer by noting that in the considered modified statistical model the specific heat of the nucleon assembly is reduced by a factor γ_0 as compared to that of a strongly degenerated gas moving in a constant potential well. It follows that the momentum dependence of the real part of the potential, introduced into the statistical model through the nucleon effective mass approximation, is responsible for the reduction of about 40% of the degrees of freedom of the nucleus as a whole. Thus Eq. (108) takes into account the nuclear interactions simply by virtue of the factor γ_0.

2.2. *The effect of two-body interactions on the nuclear excitation energy.* – For a better understanding of the modified expression (108) of the nuclear excitation energy, we shall now examine, independently of the nucleon effective mass approximation, how the explicit introduction of two-body forces into the statistical model alters the law $Q(\tau) = a\tau^2$.

On calling $\langle T(\tau)\rangle$ and $\langle V(\tau)\rangle$ the average kinetic respectively potential energy of the nucleus at the temperature τ, the excitation energy is given by

$$Q(\tau) = \langle T(\tau)\rangle - \langle T(0)\rangle + \langle V(\tau)\rangle - \langle V(0)\rangle \,, \tag{111}$$

from which the relation $Q(\tau) = a\tau^2$ is obtained assuming $\langle V(\tau)\rangle = \langle V(0)\rangle$. This happens if the potential in which the nucleons move is constant, and therefore Eq. (1) is assumed to be valid. The situation is far more complicated if the nucleon total energy is given by Eq. (2), because in this case one should expect that the real part of the potential varies with the nuclear excitation, *i.e.*

$$\mathcal{V}_R(k, \tau) = \mathcal{V}_0 f(k, \tau) \,, \tag{112}$$

reducing to Eq. (3) at the limit $\tau = 0$. If we confine ourselves to low excitation energies and make systematically use of the Fermi approximation ([32]), the variation of $\mathcal{V}_R(k, \tau)$ with excitation can be determined without the explicit knowledge of $f(k)$ at $\tau = 0$. It is in fact easy to realize that Eq. (112), up to terms in τ^2, can be written as

$$\mathcal{V}_R(k, \tau) = \mathcal{V}_0[f_0(k, \tau) + (\tau/E_F)^2 f_1(k, \tau)] \,, \tag{113}$$

([34]) E. D. COURANT: *Phys. Rev.*, **82**, 703 (1951).
([35]) R. G. BLIN-STOYLE: *Nuclear Physics*, **2**, 169 (1956).

where $f_0(k, \tau)$ and $f_1(k, \tau)$, as will be shown below, are two functions dependent on the choice of the two-nucleon potential.

The nucleus potential energy at the temperature τ reads

$$\langle V(\tau)\rangle = \frac{3A\mathcal{V}_0}{2k_F^3}\int_0^\infty k^2 g(k,\tau)[f_0(k,\tau) + (\tau/E_F)^2 f_1(k,\tau)]\,\mathrm{d}k\,. \tag{114}$$

The difference $\langle V(\tau)\rangle - \langle V(0)\rangle$ can be evaluated using the following rule, which summarizes the Fermi approximation up to terms in τ^2,

$$\int_0^\infty F(\varepsilon)g(\varepsilon,\tau)\,\mathrm{d}\varepsilon = \int_0^{\varepsilon_F(\tau)} F(\varepsilon)\,\mathrm{d}\varepsilon + \frac{\pi^2\tau^2}{6}\left[\frac{\partial F(\varepsilon)}{\partial\varepsilon}\right]_{\varepsilon=\varepsilon_F(\tau)}. \tag{115}$$

For a standard nucleus, the parameter $\varepsilon_F(\tau)$, introduced by the normalization prescription of the Fermi distribution, is obtained from Eq. (97) assuming $\gamma_0 = 1$ $(\lambda_n = \nu_n = 1)$. The corresponding momentum, up to terms in τ^2, is given by

$$k_F(\tau) = k_F[1 - (\pi^2/24)(\tau/E_F)^2]\,. \tag{116}$$

Using Eq. (115) and collecting by means of Eq. (116) all contributions to the τ^2-term, it is found

$$\langle V(\tau)\rangle - \langle V(0)\rangle = \frac{3A\mathcal{V}_0}{2k_F^3}\,[L_0(k_F)(\tau/E_F)^2 - L_1(k_F,\tau)], \tag{117}$$

where

$$L_0(k_F) = \int_0^{k_F} k^2 f_1(k,0)\,\mathrm{d}k + \frac{(2M)^{\frac{3}{2}}E_F^2}{12}\left\{\frac{\partial}{\partial\varepsilon}\,[\varepsilon^{\frac{1}{2}} f_0(\varepsilon,0)]\right\}_{\varepsilon=E_F}, \tag{118a}$$

$$L_1(k_F,\tau) = \int_0^{k_F} k^2 f(k)\,\mathrm{d}k - \int_0^{k_F(\tau)} k^2 f_0(k,\tau)\,\mathrm{d}k\,. \tag{118b}$$

Since $L_0(k_F)$ is multiplied by τ^2 and we neglect the τ^4-terms, the momentum k_F has been used in Eq. (118a) instead of $k_F(\tau)$. For the same reason the functions $f_i(k,\tau)$ $(i = 0, 1)$ have been written for $\tau = 0$ as $f_i(k, 0)$.

Taking into account Eq. (117), Eq. (111) becomes

$$Q(\tau) = a\Gamma\tau^2\,, \tag{119}$$

where

$$\Gamma(k_F,\tau) = 1 + \frac{6\mathcal{V}_0}{\pi^2 k_F^3}\left[\frac{L_0(k_F)}{E_F} - L_1(k_F,\tau)\frac{E_F^2}{\tau^2}\right]. \tag{120}$$

In order to compare Eq. (119) with Eq. (108) to see whether the function $\Gamma(k_F, \tau)$ is quantitatively equivalent to γ_0, we need an explicit expression for $\mathcal{V}_R(k, \tau)$. For simplicity's sake, we choose the two-nucleon potential (13), which led us to $\mathcal{V}_R(k) = \mathcal{V}_R(k, 0)$, given in Eq. (14). Following the method used in Sect. **1**, the real part of the nuclear potential at the temperature τ is

$$\mathcal{V}_R(k, \tau) = -\frac{9f^2 A}{4\Omega k_F^3} \int \frac{g(k', \tau)\, \mathrm{d}\boldsymbol{k}'}{\mu^2 + |\boldsymbol{k}' - \boldsymbol{k}|^2}, \tag{121}$$

i.e.

$$\mathcal{V}_R(k, \tau) = -\frac{3f^2}{2\pi k} \int_0^\infty k' g(k', \tau) \ln \frac{\mu^2 + (k' + k)^2}{\mu^2 + (k' - k)^2}\, \mathrm{d}k'. \tag{122}$$

Eq. (122) is readily evaluated by a straightforward application of the rule (115). Then, after collection of all contributions to the τ^2-terms, by repeated use of Eq. (116), it is found that $\mathcal{V}_R(k, \tau)$ has the form (113), where $f_0(k, \tau)$ is simply given by Eqs. (14) with k_F replaced by $k_F(\tau)$, and

$$f_1(k, 0) = \frac{\pi^2 k_F^2}{12\mathcal{G}(0)} \frac{\mu^2 - k_F^2 + k^2}{[\mu^2 + (k_F + k)^2][\mu^2 + (k_F - k)^2]} - \frac{\pi^2}{12} f_0(k, 0). \tag{123}$$

The calculation of Eq. (118*a*) gives

$$L_0(k_F) = (\pi^2 k_F^3/24)\xi_0(k_F), \tag{124a}$$

where

$$\xi_0(k_F) = \frac{\mathcal{V}_R(k_F)}{\mathcal{V}_0} - \frac{2}{3A} \frac{\langle V(0) \rangle}{\mathcal{V}_0} + \frac{2 + \mathcal{G}(0)}{\mathcal{G}(0)} + \frac{k_F^2 - 2\mu^2}{\mu k_F \mathcal{G}(0)} \operatorname{tg}^{-1}\left(\frac{2k_F}{\mu}\right) - \\ - \frac{1}{2}\left[\frac{1 + \mathcal{G}(0)}{\mathcal{G}(0)} + \frac{\mu^2}{2k_F^2}\right] \ln \frac{\mu^2 + 4k_F^2}{\mu^2}. \tag{124b}$$

The function $L_1(k_F, \tau)$, defined in Eq. (118*b*), can be derived from the difference between Eq. (16) and the same equation with k_F replaced by $k_F(\tau)$. Evaluating all contributions to the τ^2-terms by means of Eq. (116), it is found

$$L_1(k_F, \tau) = (\pi^2 k_F^3 \tau^2/16E_F^2)\xi_1(k_F), \tag{125a}$$

where

$$\xi_1(k_F) = 1 - \frac{\mu^2}{18k_F^2} + \frac{\mu^2}{6k_F^2}\left(1 + \frac{\mu^2}{12k_F^2}\right) \ln \frac{\mu^2 + 4k_F^2}{\mu^2} - \frac{\mu}{3k_F} \operatorname{tg}^{-1}\left(\frac{2k_F}{\mu}\right). \tag{125b}$$

The parameter Γ is thus independent of τ^2 and reads

$$\Gamma(k_F) = 1 + \frac{\mathcal{V}_0}{4E_F} [\xi_0(k_F) - \tfrac{3}{2}\xi_1(k)_F]. \tag{126}$$

Assuming $R = 1.22A^{\frac{1}{3}} \cdot 10^{-13}$ cm, from Eqs. (14) and (16) it is obtained $\langle V(0) \rangle / A = 0.36 \mathcal{V}_0$ and $\mathcal{V}_R(k_F) = 0.58 \mathcal{V}_0$. Eqs. (124*b*) and (125*b*) give $\xi_0(k_F) = 1.877$ and $\xi_1(k_F) = 0.876$, *i.e.* $\Gamma(k_F) = 1 + 3.5 \cdot 10^{-3} \mathcal{V}_0$. In order to have $B = -15$ MeV at the assumed nuclear density, irrespective of the position of the minimum of the nucleus total energy, it must be $\mathcal{V}_0 = -82.5$ MeV and finally $\Gamma(k_F) = 0.711$. It follows that

$$\Gamma \sim \gamma_0 = \frac{2M^*}{M + M^*} . \tag{127}$$

In the considered case, the excitation energy (119) with $\Gamma = 0.711$ implies a nucleon effective mass $M^* = 0.551 M$.

Eq. (127) supports the preceding remark concerning the applicability of the effective mass approximation to an excited nucleus, and shows how the complicated dependence of the function $\Gamma(k_F)$ on the strength and range of the two-body potential is simply accounted for by the parameter γ_0 and ultimately by the nucleon effective mass M^*. The result $\Gamma \sim 0.7$ is independent of the particular two-body potential and of the saturating nature of the nuclear forces, provided the value of $\mathcal{V}_R(k_F)$ and $\langle V(0) \rangle$ are chosen in agreement with the correct volume energy at the considered density. This is true only on the basis of the assumed approximation, restricted to the first order perturbation and to terms in τ^2. In the general case, the parameter Γ is strongly influenced by the odd derivatives of the nuclear potential with respect to the momentum k, evaluated at the top of the Fermi sphere. This mathematical circumstance suggests that Γ is not independent of τ and the level density parameter, evaluated from Eq. (111), should be written as $a^* = a\Gamma(k_F, \tau)$. The consequences of this fact may well explain the dependence of the level density parameter on the excitation energy, as suggested by HAYAKAWA and KIKUCHI ([36]), the values $a > a^*$, consistent with Gugelot's experiments ([37]), and the surprinsing tendency of the nuclear temperature to increase with excitation more slowly than expected, or eventually to decrease as pointed out by COHEN ([38]). In this way, a theoretical correlation could be established between the direct measurements of the level density at excitations energies corresponding to slow neutron capture (Sect. 2·5) and the measurements of the energy spectra of evaporated particles (Sect. 2·6). In the transition region between small excitation energies, where the shell model is applicable and the level density parameter is strongly influenced by the nucleon effective mass ratio γ, and large excitations, where the statistical model is valid, this theory should reveal physical features reflecting characteristics of both types of model.

([36]) S. HAYAKAWA and K. KIKUCHI: *Progr. of. Theor. Phys.*, **12**, 574 (1954).
([37]) P. C. GUGELOT: *Phys. Rev.*, **81**, 51 (1951).
([38]) B. I. COHEN: *Phys. Rev.*, **42**, 1245 (1953).

2·3. *The general features of the modified statistical model of the nucleus.* – The basic features of the statistical model, modified according to the nucleon effective mass approximation, ultimately are: (*a*) the maximum momentum $k_F = (2M\gamma_0 E_F^*)^{\frac{1}{2}}$ allowed to the nucleons is equal to that which is expected according to the conventional model ($\gamma_0 = 1$); (*b*) the new Fermi energy $E_F^* = \gamma_0^{-1} E_F$ is larger than E_F (incidentally we point out that this fact strongly improves the Fermi approximation); (*c*) the level density parameter $a^* = a\gamma_0$ is smaller than the parameter $a = \pi^2 A/4E_F$ of the conventional model. These peculiar features are identical to those of the so called « correlated model » ([39]). It follows that the ratio $\gamma_0 = 0.7$ and the underlying idea of the nucleon effective mass approximation is implicit in Bardeen's pioneering works on heavy nuclei.

Another conclusion which can be drawn from our statistical model concerns the final result of Watanabe's theory ([40]). Since the attractive nature of the potential $\mathcal{V}_R(k)$ implies, according to Eq. (23), $\gamma_0 < 1$, the zero point energy $W_0 = (3/5)AE_F$ of the conventional model is smaller than W_0^*, following, for a standard nucleus, from Eq. (105). Thus, being $W_0^* > W_0$, the difference $\Delta W = W_0^* - W_0$ may be regarded as thermal agitation energy brought about by the momentum dependence of $\mathcal{V}_R(k)$ and corresponding to a certain nuclear temperature τ_0. This temperature τ_0 is easily determined ([41]) by equating the difference ΔW to the excitation energy of the conventional model. For a standard nucleus it is found.

$$\tau_0 = (12/5)^{\frac{1}{2}}(E_F/\pi)[(1-\gamma_0)/\gamma_0]^{\frac{1}{2}} . \tag{128}$$

The value $\tau_0 \simeq 7.7$ MeV, obtained from Eq. (128) for $\gamma_0 = 0.7$, and $r_0 = 1.22 \cdot 10^{-13}$ cm, is very close to that empirically assumed by HEIDEMANN ([42]) to describe the internal momentum distribution of heavy nuclei, in satisfactory agreement with York's data on deuteron formation by pick-up. We emphasize that the result stated in Eq. (128) can be compared with Watanabe's theory only in so far it shows that the mean kinetic energy of the nucleons bound in the nucleus is higher than the kinetic energy of non-interacting nucleons, confined in the nuclear volume Ω. The underlying idea of Watanabe's theory is however different from that on which the nucleon effective mass approximation is based. In fact, the former theory assumes that when the interactions between the nucleons are switched on, their effect consists in transitions between occupied and unoccupied states. Because of the Pauli principle, these transitions will spread

([39]) J. BARDEEN: *Phys. Rev.*, **61**, 799 (1973); ref. ([32]), p. 85.
([40]) S. WATANABE: *Zeits. f. Phys.*, **113**, 482 (1939).
([41]) L. ROSENFELD: *Nuclear Forces* (Amsterdam, 1948), p. 253.
([42]) J. HEIDEMANN: *Phys. Rev.*, **80**, 171 (1950).

the unperturbed distribution in the neighborhood of the limiting momentum k_F, lowering the density in momentum space below k_F and giving rise to a tail in the distribution above this value. As has been previously pointed out, our statistical model, based on the first order perturbation, does not rise any problem concerning the fractional occupancy of phase states, and neither the perturbed Watanabe distribution nor the Heidmann one, ruled by $g(k, \tau_0)$, or any other involving high momentum components, can be derived from the present formulation of the model. Althought the contributions to $\langle V(\tau) \rangle$ due to the second order perturbation might be essential for the correct evaluation of the temperature dependence of the excitation energy according to Eq. (111), it is important to stress that, because of the assumed momentum dependence of the nuclear potential, our statistical model at $\tau = 0$ exhibits several features which otherwise are characteristic of the second order perturbation. In fact, one of the consequences involved by any internal momentum distribution, deviating from the rectangle-like one, is that the mean kinetic energy per nucleon is higher than $(3/5)E_F$. This is just the effect which is brought about by the momentum dependence of the nuclear potential, because the new Fermi energy E_F^* comes out to be larger than E_F. According to the nucleon effective mass approximation, the mean kinetic energy per nucleon is $\langle \varepsilon \rangle = (3/5)(E_F/\gamma_0)$ *i.e.* about 30 MeV for $\gamma_0 = 0.7$ and $r_0 = 1.22 \cdot 10^{-13}$ cm. This value is intermediate between that following from the Watanabe or Heidemann (~ 23 MeV) and the Wilcox distribution ([43]) (~ 33 MeV), recently used for fitting quasi-elastic scattering data. Furthermore, the number of nucleons, which, according to the present model, have an energy larger than E_F is given by $A(1 - \gamma_0^3)$, *i.e.*, for $\gamma_0 = 0.7$, ($r_0 = 1.22 \cdot 10^{-13}$ cm) about 45% of the total number A of nucleons is above the Fermi energy of the conventional model. This is another aspect of our model which is characteristic of an internal momentum distribution. Because of these peculiar features, a re-examination, in the light of the momentum dependence of the nuclear potential, of stripping and pick-up processes, quasi-elastic collisions and meson production in nucleon-nucleus collisions, the interpretation of which is based on the existence of high momentum components, would be not without interest for a better understanding of the essential role of the internal momentum distribution in fitting the data. In fact, since in this kind of analyses the momentum distribution of the struck nucleon is fixed by energy-momentum conservation and the target nucleons are considered in motion in a constant potential well, the suspicion arises that in some cases the assumption of high momentum components might lead to the same results otherwise obtainable by taking into account the momentum dependence of the nuclear potential.

([43]) J. M. WILCOX: *UCRL*-3475 (Berkeley, 1956).

2·4. *The nucleus ground state. Determination of the nucleon effective mass from the symmetry energy.* – A direct information on the value of γ_0 and therefore on the value of the nucleon effective mass M^* is easily obtained from the experimental coefficient a_3 of the mass correction term

$$\Delta B_3 = -a_3(A-2Z)^2/A\,, \tag{129}$$

appearing in the Weiszäcker formula. The energy ΔB_3 can be evaluated as the difference between the total energy of Z protons and N neutrons and the minimum value of the nucleus total energy, which occurs for $Z=N=A/2$. On the basis of the nucleon effective mass approximation it is found

$$\Delta B_3 = W^*_{0\,(\min)} - \sum_{i=1,2} W^*_0(N_i)\,, \tag{130}$$

i.e., using Eq. (99) with $\gamma_0 = \text{const}$ ($\lambda_n = \nu_n = \gamma_0^{\frac{3}{2}}$) and introducing the parameter $\delta = A - 2Z$, it is found

$$\Delta B_3 = -(3E_F/10\gamma_0)A^{-\frac{2}{3}}\Big\{\sum_{i=1,2}[A+(-1)^{i+1}\delta]^{\frac{5}{3}} - 2A^{\frac{5}{3}}\Big\}\,. \tag{131}$$

The Taylor expansion, as far as terms in δ^2 of the two binomials appearing in Eq. (131), transforms this equation into Eq. (129), where the coefficient a_3 reads

$$a_3 = E_F/3\gamma_0\,. \tag{132}$$

Assuming the experimental value $a_3 = 19.3$ MeV, Eq. (132) gives $\gamma_0 = 0.570$ for $R = 1.22A^{\frac{1}{3}}\cdot 10^{-13}$ cm, *i.e.* $M^* \sim 0.4M$. It follows that the conventional statistical model is capable of giving only the dependence of the symmetry energy on $(A-2Z)^2/A$, but the coefficient a_3 must be empirically determined, since its calculated value ($a_3 = E_F/3 = 10.9$ MeV) is in disagreement with nuclear stability. From Eq. (132) it is also found

$$r_0^2\gamma_0 = \frac{(9\pi)^{\frac{2}{3}}}{24Ma_3} = 0.839\cdot 10^{-22}\ \text{cm}^2\,, \tag{133}$$

from which a simple relation may be derived between the nucleon effective mass and the nuclear density.

2·5. *The excited nucleus. Determination of the nucleon effective mass from the level density.* – It is known that a rough description of the nuclear level density $\omega(Q) = [D_0(Q)]^{-1}$, where $D_0(Q)$ is the level spacing of all spins and

parities, is obtained from the simple relation

$$\omega(Q) = c \exp [2(aQ)^{\frac{1}{2}}] , \tag{134}$$

where the parameters a and c are fixed from the spacing of levels measured from the resonance capture of slow neutrons ($Q \sim 6-8$ MeV) respectively from the observed level density at very low excitation ($Q \sim 1$ MeV). This relations is clearly incapable of reproducing details concerning individual nuclei, except that it qualitatively accounts for the larger level separation experimentally observed in magic-plus-one nuclei, because the lower binding energy of the last nucleon implies a lower excitation energy Q. The exponential dependence on Q, given by Eq. (134), follows from general thermodynamical considerations ([44,45]), which show how the parameters a and c depend on the assumed nuclear model. In fact, on calling $S(Q)$ the entropy of the strongly degenerated nucleon gas, Eq. (134) follows from the general expression

$$\omega(Q) = \frac{\exp [S(Q)/\tau]}{\tau[2\pi(\mathrm{d}Q/\mathrm{d}\tau)]^{\frac{1}{2}}} , \tag{135}$$

using for Q the relation $Q(\tau) = a\tau^2$. In this case it is found $c = (4\pi)^{-\frac{1}{2}}$ and, as we already know, $a = \pi^2 A/4E_F$. Using the values of a and c, computed from these two relations, the calculated level spacing is in complete disagreement with experiments (Fig. 3, $\gamma_0 = 1$). This is the reason why the use of Eq. (134) requires the empirical determination of a and c with no reference at all to the values calculated on the basis of the statistical model. This situation closely recalls the one men-

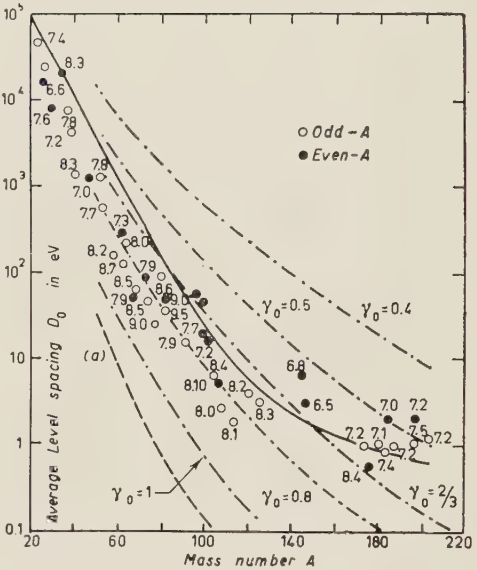

Fig. 3. – Average spacing $D_0(Q)$ of excited levels in «non-magic» nuclides (ref. ([48])). The actual excitation energy for each compound nucleus is shown numerically beside the individual points. The solid curve represents the level spacing, computed from Eq. (134), with $Q = 8$ MeV and Weisskopf's evaluation of a and c from independent experimental data. The dotted curves have been plotted using Eq. (136) with $Q = 8$ MeV and $a^* = a\gamma_0$ ($R = 1.22\, A^{\frac{1}{3}} \cdot 10^{-13}$ cm). The curve (a) represents $D_0(Q)$ with $\gamma_0 = 1$ and $r_0 = \mu^{-1}$.

([44]) J. M. Blatt and V. F. Weisskopf: *Theoretical Nuclear Physics* (New York, 1952), p. 365.

([45]) I. N. Sneddon and B. F. Touschek: *Proc. Phil. Soc.*, **44**, 39 (1948).

tioned in Sect. 2·4, in connection with the coefficient a_3 of the symmetry energy. The failure of the conventional model ultimately depends on the assumption that Eq. (1) is valid ($\gamma_0 = 1$). The level spacing $D_0(Q)$ becomes larger than in the case $\gamma_0 = 1$ if the momentum dependence of the nuclear potential is taken into account, because the total energy of the equivalent particle motion increases more strongly with the momentum than the kinetic energy alone: the phase volume available for levels is increased and the individual particle states are reduced as well as the density of levels of the whole system. Introducing into Eq. (135) the excitation energy given by Eq. (106), it is found

$$D_0(Q) = (4\pi)^{\frac{1}{2}}(Q^3/a^*)^{\frac{1}{4}} \exp\left[-2(a^*Q)^{\frac{1}{2}}\right]. \tag{136}$$

The values of $D_0(Q)$, computed from Eq. (136) for $Q = 8$ MeV and for different values of γ_0, are compared with the experimental data in Fig. 3. It is seen that for $\gamma_0 = 0.66$ ($M^* \sim 0.5M$) the general trend of the level spacing is fairly well reproduced. We remark that for $\gamma_0 = 0.66$ ($R = 1.22A^{\frac{1}{3}} \cdot 10^{-13}$ cm) the values of the modified level density parameter a^* are very close to those empirically used for fitting the level spacing ([46]) or the photodisintegration of heavy nuclei ([47]). Inspection of Fig. 3 shows that the agreement with the data would be greatly improved by varying γ_0 from 0.80 for $A \sim 60$ to 0.5 for $A \sim 200$. This variation of γ_0 follows directly from the radial dependence of the nuclear potential, as is readily recognized by noting that, on the basis of the radial effective mass approximation, the level spacing is obtained using Eq. (109) into Eq. (136). The function $l(r)$, derived from Eq. (77*b*), increases with increasing nuclear radius and therefore $\gamma(r)$, defined in Eq. (83), decreases with R. It is then easily proved that the average value $\langle\gamma_0\rangle$, appearing in Eq. (109), is a decreasing function of R, *i.e.* of the atomic mass number A, as is required to improve the fit of the data shown in Fig. 3. We may, therefore, conclude that the current use of Eq. (134) with the empirically adjusted values of a have its logical theoretical justification in the nucleon effective mass concept.

For illustrative purposes the level spacing of 29 nuclides ([48]) has been fitted using Eq. (136) (Table VI). Since a^* is proportional to r_0^2, the value of the nucleon effective mass can be increased by decreasing slightly r_0. It is interesting to compare the values of r_0^2, listed in Table VI, with that computed from the symmetry energy [Eq. (133)]. Inspection of Table VI shows that, apart from some rapid fluctuations of the value of a^*, due to the even-odd

([46]) Ref. ([44]), p. 372.
([47]) B. C. DIVEN and G. M. ALMY: *Phys. Rev.*, **80**, 407 (1950).
([48]) D. J. HUGHES, R. C. GARTH and I. S. LEVINE: *Phys. Rev.*, **91**, 1423 (1953).

TABLE VI. – *Determination of M^*/M from the level spacing $D_0(Q)$ using Eq.* (136). *The values of γ_0 and of M^*/M listed in the last two columns have been determined using* $R = 1.22 A^{\frac{1}{3}} \cdot 10^{-13}$ cm.

Nucleus	A	Z	Q (MeV)	D_0 (eV)	a^* $(\text{MeV})^{-1}$	$r_0^2\gamma_0 \cdot 10^{26}$ cm^{-2}	γ_0	M^*/M
Kr	86	36	6.17	$3.1 \cdot 10^2$	4.32	0.986	0.662	0.495
Rb	85	37	0.59	35.0	4.35	1.003	0.674	0.508
Rb	87	37	6.67	$4.1 \cdot 10^2$	3.83	0.864	0.581	0.409
Sr	88	38	7.12	$3.8 \cdot 10^2$	3.70	0.825	0.554	0.383
Y	89	39	7.63	$1.1 \cdot 10^2$	4.36	0.962	0.646	0.477
Nb	93	41	7.94	16.0	5.67	1.096	0.736	0.582
Mo	98	42	7.32	58.0	5.02	1.005	0.672	0.506
Mo	100	42	6.87	47.0	5.44	1.068	0.676	0.510
Ru	102	44	7.67	18.0	5.75	1.105	0.742	0.590
Ru	104	44	7.21	17.0	6.12	1.143	0.767	0.622
Ag	107	47	8.41	6.0	6.21	1.139	0.765	0.691
Sn	124	50	7.38	19.8	5.90	0.933	0.627	0.457
Sb	121	51	8.18	4.2	5.67	1.082	0.727	0.571
I	127	53	8.33	3.3	6.75	1.043	0.701	0.540
La	139	57	6.55	63	5.44	0.768	0.516	0.348
Ce	140	58	8.85	52	5.40	0.757	0.508	0.340
Ce	142	58	6.54	65	5.42	0.749	0.503	0.330
Pr	141	59	7.15	26	5.79	0.806	0.541	0.371
Nd	146	60	6.78	6.7	7.34	0.986	0.662	0.549
Nd	148	60	6.46	3.3	8.42	1.117	0.750	0.600
Lu	175	71	7.19	1.1	8.83	0.990	0.665	0.498
Lu	176	71	8.42	0.5	8.49	0.847	0.636	0.466
Ta	181	73	7.12	1.13	8.86	0.960	0.645	0.476
W	186	74	7.01	21.8	8.29	0.864	0.580	0.408
Re	185	75	7.40	0.87	8.83	0.937	0.629	0.459
Re	187	75	7.23	0.93	8.97	0.941	0.632	0.462
Pt	198	78	7.16	2.14	8.15	0.907	0.609	0.438
Au	197	79	7.52	1.15	8.36	0.833	0.559	0.389
Hg	204	80	7.19	1.26	8.66	0.833	0.559	0.389

character of the nucleus, which is ignored in the present formulation of the theory, the variation of M^*/M between 0.4 and 0.6 is well established.

2.6. *Effect of the momentum dependence of the nuclear potential on the evaporation spectra.* – It is known that several experiments seem to disagree with the idea that the compound nucleus de-excites itself by evaporating nucleons according to a maxwellian energy distribution. The data on which this evidence is based indicate that there are more high energy nucleons emitted than can be explained by a maxwellian distribution. The excess of high energy nucleons is « observed » by comparing the measured energy spectrum with the current

formula

$$I(E)\,\mathrm{d}E = \text{const.}\; E\sigma_c(E)\exp\left[-(E/\tau)\right]\mathrm{d}E\,. \tag{137}$$

It is generally believed that this experimental result conflicts with the evaporation theory and gives evidence of a direct interaction mechanism. The extensive evidence that the simple statistical theory with $\gamma = \gamma_0 = 1$ does not adequately represents the observed facts suggests that the influence on the compound nucleus of the momentum dependence of the nuclear potential is probably not negligible. This point should be carefully examined before drawing any conclusion about the suspected failure of the evaporation theory in explaining the shape of the energy spectrum. The preconceived idea, implicit in Eq. (137), that the entropy of the nucleon gas is a linear function of the nuclear temperature, is certainly valid for an « ideal » gas, because in this case the contribution of the τ^4-term to the nuclear excitation is entirely negligible. Whether this is still true for a « real » nucleon gas, whose excitation energy is evaluated according to Eq. (111), is an open problem. The answer to this question could be given by extending the procedure outlined in Sect. 2·2 to the second order perturbation and to the τ^4-term.

The influence of the momentum dependence of the potential on the evaporation spectra is readily recognized even on the basis of the nucleon effective mass approximation, which admittedly is insufficient to explain many of the physical features of the de-excitation process. On calling E the energy of the emitted particle and E_0 the maximum energy at which the residual nucleus can be excited, the energy spectrum evaluated in term of Weisskopf's theory, using Eq. (108), turns out to be

$$I(E)\,\mathrm{d}E = \frac{(2s+1)M}{\pi^2}\,\sigma_c(E)E\exp\left\{2(a^*E_0)^{\frac{1}{2}}\left[\left(1-\frac{E}{E_0}\right)^{\frac{1}{2}}-1\right]\right\}\mathrm{d}E\,, \tag{138}$$

where s is the spin of the outgoing particle. Eq. (138) reduces to Eq. (137) for $\gamma_0 = \gamma = 1$ and $E \ll E_0$. The modified energy spectrum possesses a maximum which is shifted towards relatively higher energies and increased, as compared to that of Eq. (137), approximately by a factor $\gamma_0^{-\frac{1}{2}}$. Furthermore, at the end of the spectrum the shape is evidently non-maxwellian. The plot of $\ln I(E)/E$ against $2E_0[(1-E/E_0)^{\frac{1}{2}}-1]$ is a straight line, whose negative slope is the inverse of the nuclear temperature. In this way it is possible to derive the nucleon effective mass directly from the evaporation spectra. Eq. (138) shows that the dispersive nature of the nuclear potential produces a high energy tail of the evaporated particles, which is practically suppressed in the case $\gamma_0 = 1$. Therefore, one should expect that, for not too high excitation energies, the evaporation processes are more important than usually believed, and that at least a part of the high energy nucleons observed in

the emission spectra have an evaporative origin. The following physical argument will support this conclusion and qualitatively clarify the problem.

An intuitive distinction between evaporative and direct processes can be made according to the value of the ratio v_N/v_S, where v_N is the velocity of the incoming nucleon inside the nucleus and v_S is the velocity of sound in nuclear matter. If the nuclear « Mach number » v_N/v_S is smaller than 1, the elastic tide, due to the entrance of the nucleon in the nucleus, shares the available energy among all degrees of freedom of the nucleus, before the nucleon has passed through it. This situation is characteristic of evaporative processes, which require the thermodynamical equilibrium to be reached before the nucleus undergoes the de-excitation. The opposite case ($v_N/v_S > 1$) is clearly characteristic for the direct interaction. Since the velocity of the elastic waves in nuclear matter is proportional to $E_F^{\frac{1}{2}}$ and inversely proportional to the square root of the nucleon mass M ([19]), it follows that the value v_S^*, consistent with the nucleon effective mass approximation, is related to the value v_S, calculated according to the conventional model, by the relation $v_S^* = \gamma_0^{-1} v_S$, *i.e.* for $\gamma_0 = \frac{2}{3}$ the velocity of sound in the « real » nucleon gas has a value higher by a factor 1.5 than the velocity in an « ideal » nucleon gas. Thus, for a given nucleon velocity v_N, the momentum dependence of the nuclear potential reduces the nuclear « Mach number » v_N/v_S and therefore increases the contribution of the evaporative processes to the nuclear de-excitation.

* * *

The Authors wish to place on record their sincere appreciation to Professor L. Rosenfeld for his interest in this paper and for invaluable criticism.

([19]) Ref. ([41]), p. 250.

RIASSUNTO

Il modello statistico del nucleo viene riesaminato tenendo conto della dipendenza della parte reale del potenziale nucleone-nucleo dalla velocità del nucleone. Le modifiche così introdotte nel modello vengono discusse con particolare riguardo al concetto di massa effettiva del nucleone, massa che viene determinata sia valendosi del valore sperimentale dell'energia di simmetria, sia usando la densità dei livelli nucleari ottenuta nelle esperienze di cattura con neutroni lenti.

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

An Explicit Solution of the Thirring Model.

V. Glaser (*)

CERN, Theoretical Study Division - Geneva

(ricevuto il 23 Maggio 1958)

Summary. — The two-dimensional model of a relativistic theory proposed recently by W. Thirring (1) is solved by displaying the field operator ψ as an explicit functional of the corresponding incoming field. The S-matrix (without external sources) is found to be of the form $S=\exp[iQ_1Q_2]$, where $Q_{1,2}$ are two constants of the motion. The S-matrix is unitary, but gives rise only to a relative change of phase of the plane waves associated with the colliding particles, all the cross-sections being equal to zero (2). After the renormalization, all the matrix elements of the field operator turn out to be finite analytic functions of the coupling constant. An apparent discrepancy with the results of Thirring (1) is discussed.

1. – Introduction.

In a very recent paper (1) W. Thirring has proposed to study the universal field equation of Heisenberg (3) $\boldsymbol{p}\psi+2g(\bar{\psi}\psi)\psi=0$ in the special case of one space dimension (x) and one time dimension (t) (+). By using a very interesting method Thirring was indeed able to calculate the exact expressions for the matrix elements $\langle 1|\psi(x)|0\rangle$ and $\langle 3|\psi(x)|0\rangle$, $|1\rangle$, $|3\rangle$ being the one- and the three-particle states respectively and $|0\rangle$ the vacuum state, as func-

(*) On leave of absence from the Institute « Rudjer Bošković », Zagrèb.

(1) W. E. Thirring: *Ann. Phys.*, **9**, 91 (1958).

(2) Compare, however, W. E. Thirring: *Nuovo Cimento*, **9**, 1007 (1958).

(3) W. Heisenberg: *Zeits. f. Naturfor.*, **9***a*, 292 (1954).

(+) There is, as noted by Thirring, an important difference between these two cases: in the four-dimensional theory the coupling constant g has the dimension of a square of a length, while in the two-dimensional model it is dimensionless.

tions of the momenta and of the coupling constant. He also suggested that his two-dimensional model was very probably completely soluble.

In this paper we shall give a complete solution of the Thirring model by applying more straight forward methods. We shall be able to express the field operator ψ as an explicit functional of the incoming field. After an infinite field renormalization all the matrix elements of ψ will turn out to be finite analytical functions of the coupling constant g. The S-matrix will be shown to be unitary and of a particularly simple form: $S = \exp[igQ_1Q_2]$, where Q_1 and Q_2 are two constants of the motion. The only effect of the collision is a relative change of phase of the plane waves after their mutual penetration (2).

Furthermore, it will be shown that, due to an ambiguity in the process of solving the basic equation in the approach of THIRRING, our results differ from those of THIRRING in as far as the coupling constant $\lambda = g/2$ in his final formula is replaced by $\operatorname{tg} g/2$. This contradiction, however, can be resolved if one adopts the point of view that all the observable quantities of the theory should be actually expressed not as functions of the unobservable coupling constant but as functions of a fixed arbitrarily chosen observable quantity (*).

In this case the question whether observable quantities can be expanded in a power series in g with a *finite* or *infinite* radius of convergence seems to become irrelevant. We believe, however, that our field theoretic approach represents a fairly unique way to avoid the ambiguities of the approach of THIRRING.

The next section is devoted to the solution of the equations of motion and the commutation relations. In the third section a comparison with the approach of THIRRING is made. The last section is devoted to the calculation of the S-matrix and to the renormalization of the field-operators. The appendix contains the discussion of an integral equation.

2. – Solution of the equations of motion.

We shall start from the field-equation

$$-i\gamma^\mu \partial_\mu \psi(x, t) + 2g(\overline{\psi}\psi)\psi(x, t) = 0 \quad (+)\,, \tag{1}$$

where $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$, $\overline{\psi} = \psi^*\beta$, $\beta = \lambda^2$. According to THIRRING (1) it is con-

(*) I am indebted to Dr. THIRRING for this remark, private communication. Compare (5).

(+) Units: $\hbar = c = 1$. Matrix tensor: $g^{11} = -g^{22} = 1$, $g^{12} = g^{21} = 0$, the index 1 referring to the space co-ordinate x, the index 2 to the time co-ordinate t. g is connected to the coupling constant λ of THIRRING's paper (1) by $\lambda = g/2$.

2813

venient to take the following representation of the Dirac matrices:

$$\gamma^1 = i\sigma_1\,, \qquad \gamma^2 \equiv \beta = \sigma_2\,, \qquad \beta\gamma^1 \equiv \alpha = \sigma_3 \tag{2}$$

where σ_ν are the Pauli matrices in their usual representation (with this choice charge-conjugation is equivalent to hermitian-conjugation). (1) must be supplemented with the anti-commutation relations

$$\{\psi_\tau(x,t),\, \psi^*_{\tau'}(x',t)\} = \delta_{\tau\tau'}\,\delta(x-x')\,, \qquad \{\psi_\tau(x,t),\, \psi_{\tau'}(x',t)\} = 0\,. \tag{3}$$

$\tau = 1, 2$ representing the spinor index.

Besides the energy-momentum vector

$$P^\mu = \int T^{\mu\nu}\,\mathrm{d}\sigma_\nu = -i\int_{-\infty}^{+\infty}\overline{\psi}\{\gamma^2\partial^\mu + \tfrac{1}{2}g^{\mu 2}\gamma^\nu\partial_\nu\}\psi\,\mathrm{d}x\,, \tag{4}$$

there exist, in this case, as pointed out by THIRRING, two other important constants of motion

$$Q_\tau = \int_{-\infty}^{+\infty}\psi^*_\tau\,\psi_\tau\,\mathrm{d}x,\ (\tau = 1, 2)\ (*), \tag{5}$$

$Q_1 + Q_2 = Q$ being the total charge.

Having thus fixed the notation, let us multiply the equation (1) by β and rewrite it in the following form

$$\begin{cases} \dfrac{\partial}{\partial u}\,\psi_1 - ig\psi^*_2\,\psi_2\psi_1 = 0\,, \\[2ex] \dfrac{\partial}{\partial v}\,\psi_2 + ig\psi^*_1\,\psi_1\psi_2 = 0\,. \end{cases} \tag{1'}$$

Here we have used the representation (2), introduced new variables $u = x+t$, $v = x-t$, and made use of the fact that $\psi^2_\tau = 0$, $\psi_1\psi_2 = -\psi_2\psi_1$ according to (3). It is now easy to see that the system (1′)—treated as a c-number system of differential equations—has the general solution

$$\begin{cases} \psi_1 = \varphi_1(v)\,\exp\left[ig\displaystyle\int_{u_0}^{u}\varphi^*_2(u)\varphi_2(u)\,\mathrm{d}u\right], \\[2ex] \psi_2 = \varphi_2(u)\,\exp\left[ig\displaystyle\int_{v_0}^{v}\varphi^*_1(v)\varphi_1(v)\,\mathrm{d}v\right], \end{cases} \tag{6}$$

(*) In the 2-dimensional case the equation (1) is equivalent to $\boldsymbol{p}\psi + g'(\overline{\psi}\gamma_\mu\psi)\gamma^\mu\psi = 0$ owing to (3). This equation gives rise to two conservation laws $\partial_\mu\overline{\psi}\gamma^\mu(1\pm\gamma_5)\psi/2 = 0$. In the representation (2) this leads to (5) because of $\gamma_5 = \alpha = \sigma_3$

where $\varphi_{1,2}$ are two arbitrary functions of v resp. u. We shall put $u_0 = -\infty$, $v_0 = +\infty$, so that for $t \to -\infty$ the solution $\psi_{1,2}$ foes over asymptotically into $\varphi_{1,2}$, *i.e.* into the interaction-free solution of (1′). Furthermore, following the procedure of YANG and FELDMAN (4) and KÄLLÉN (5), we shall replace tentatively $\varphi_{1,2}$ by the corresponding incoming field operators, which satisfy besides the free-field equations of motion the following commutation relations

$$(7) \qquad \{\varphi_1(v), \varphi_1^*(v')\} = \delta(v - v'), \quad \{\varphi_2(u), \varphi_2^*(u')\} = \delta(u - u'), \quad \{\varphi_\tau, \varphi_{\tau'}'\} = 0 .$$

We shall also need the Fourier representation of these operators:

$$(8) \qquad \begin{cases} \varphi_{1,2} = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} c_{1,2}(p) \exp [ipv, u] \, \mathrm{d}p , \\ \{c_\tau(p), c_\tau^*(p')\} = \delta_{\tau\tau'} \, \delta(p - p') , \qquad \{c_\tau(p). \, c_{\tau'}(p')\} = 0 , \end{cases}$$

the connection with the annihilation operators of a particle ($a(p)$) and of an anti-particle ($b(p)$), both of mass zero, momentum p and (positive) energy $E = |p|$, being the following:

$$(9) \qquad \begin{cases} c_{1,2}(p) = \theta_\pm(p) a(p) + \theta_\mp(p) b^*(-p) , \\ \{a(p), a^*(p')\} = \{b(p), b^*(p')\} = \delta(p - p') \quad \text{etc} . \end{cases}$$

Here $\theta_+(p) = 1$ for $p > 0$, $= 0$ for $p < 0$, $\theta_-(p) = \theta_+(-p)$. This corresponds to the standard spinor form:

$$(10) \qquad \begin{cases} \varphi(x, t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{+\infty} u(p) \{a(p) \exp [i(px - |p|t)] + \\ \qquad\qquad + b^*(p) \exp [-i(px - |p|t)]\} \mathrm{d}p , \\ u(p) = \begin{pmatrix} \theta^+(p) \\ \theta^-(p) \end{pmatrix} . \end{cases}$$

(compare ref. (1), eq. (2.25)).

It remains to be seen whether the above substitution is still compatible with the operator equations of motion (1′), and, what is still more important, with the commutation relations (3). We shall immediately verify that this

(4) C. N. YANG and D. FELDMAN: *Phys. Rev.*, **79**, 972 (1950).
(5) G. KÄLLÉN: *Ark. f. Fys.*, **2**, 187, 371 (1950).

is indeed so. For this purpose let us introduce the abbreviations:

$$(11)\qquad \begin{cases} \psi_1(x,t) = \varphi_1(v)\exp[igQ_2(u)] \\ \\ \psi_2(x,t) = \varphi_2(u)\exp[igQ_1(v)] \end{cases} \quad \text{where} \quad \begin{cases} Q_1(v) = \int_v^{+\infty} \varphi_1^* \varphi_1 \mathrm{d}v\,, \\ \\ Q_2(u) = \int_{-\infty}^{u} \varphi_2^* \varphi_2 \mathrm{d}u\,. \end{cases}$$

First of all it follows trivially from the commutation relations (7) and the hermiticity of $Q_{1,2}$ that

$$(12)\qquad \begin{cases} \psi_1^*(x,t)\psi_1(x,t) = \varphi_1^*(v)\varphi_1(v) \equiv \varrho_1(v)\,, \\ \psi_2^*(x,t)\psi_2(x,t) = \varphi_2^*(u)\varphi_2(u) \equiv \varrho_2(u)\,, \end{cases}$$

$$(12')\qquad [\varrho_1(v), \varrho_1(v')] = [\varrho_2(u), \varrho_2(u')] = [\varrho_1(v), \varrho_2(u')] = 0\,,$$

for all values of the arguments. (9′) has the important consequence that the exponentials may be differentiated and multiplied as c-number quantities. Furthermore we note the identities

$$(13)\qquad \begin{cases} \exp[-igQ_1(v)]\varphi_1(v')\exp[igQ_1(v)] = \exp[ig\theta_-(v-v')]\varphi_1(v') = \\ \qquad = \{1 + \theta_-(v-v')(\exp[ig]-1)\}\varphi_1(v')\,, \\ \exp[-igQ_2(u)]\varphi_2(u')\exp[igQ_2(u)] = \exp[ig\theta_+(u-u')]\varphi_2(u') = \\ \qquad = \{1 + \theta_+(u-u')(\exp[ig]-1)\}\varphi_2(u')\,, \end{cases}$$

which can be verified, say, by showing that the left and right hand sides satisfy the same first order differential equation in the parameter g and the same initial condition for $g = 0$. These relations make it possible to write down the commutation relations as follows

$$(14)\qquad \begin{cases} \{\psi_{\frac{1}{2}}(x,t), \psi_{\frac{1}{2}}(x',t)\} = \\ \qquad = \{\varphi_{\frac{1}{2}}(x \mp t), \varphi_{\frac{1}{2}}(x' \mp t')\}\exp[ig(Q_{\frac{2}{1}}(x \pm t) + Q_{\frac{2}{1}}(x' \pm t'))] = 0\,, \\ \{\psi_{\frac{1}{2}}^*(x,t), \psi_{\frac{1}{2}}(x',t')\} = \\ \qquad = \{\varphi_{\frac{1}{2}}^*(x \mp t), \varphi_{\frac{1}{2}}(x' \mp t)\}\exp[ig(Q_{\frac{2}{1}}(x \pm t) - Q_{\frac{2}{1}}(x' \pm t'))]\,. \end{cases}$$

For $t = t'$ the second expression reduces to $\delta(x - x')$ as it should. We compute the mixed commutator by the help of the formula:

$$(14')\qquad \{\psi_1, \psi_2'\} = \varphi_1(e_2\varphi_2' e_2^{-1})e_2 e_1' + \varphi_2'(e_1'\varphi_1 e_1'^{-1})e_1' e_2\,,$$

where the primes stand for the primed variables and e_i for the exponential with Q_i in the exponent. By the help of (13) (with $g \to -g$) this reduces to

$$(14'') \quad \{\psi_1, \psi_2'\} = \{\varphi_1, \varphi_2'\} \exp[-ig\theta_-(v'-v)] + \\ + \{\theta_+(x-x'+t-t') - \theta_+(x-x'+t'-t)\} \cdot (\exp[-ig]-1)\varphi_1\varphi_2' e_1' e_2 .$$

This expression vanishes for $t = t'$. Similarly we get:

$$(14''') \quad \{\psi_1^*, \psi_2'\} = \{\varphi_1^*, \varphi_2'\} \exp[ig\theta_-(v'-v)] e_1' e_2^{-1} + \\ + \{\theta_+(x-x'+t-t') - \theta_+(x-x'+t'-t)\}(\exp[ig]-1)\varphi_1^* \varphi_2 e_1' e_2^{-1} ,$$

which vanishes for $t = t'$.

The verification of the equations of motion is still more straight forward since the exponentials may be differentiated as ordinary functions. We leave it to the reader.

In order to see the relativistic covariance of the solution (11) we shall cast it into a covariant form. With the help of the retarded free-particle Green's function ($\hat{x} = (x, t)$, $\hat{p} = (p^1, p^2)$, $\hat{x}^2 = x^2 - t^2$, etc.)

$$(15) \qquad S_R(\hat{x}) = \frac{1}{2\pi} \int \frac{\exp[i\hat{p}\hat{x}]}{\hat{p}_+} \, \mathrm{d}^2 p = 2\pi i \beta \theta^+(t) \, \delta(x + \alpha t) ,$$

and the projection operators

$$P_{1,2} = \tfrac{1}{2}(1 \pm \gamma_5) , \qquad \gamma_5 = \gamma^2 \gamma^1 = \alpha ,$$

(11) can be written, as is easily verified, in the covariant spinor form

$$(16) \qquad \psi(\hat{x}) = P_1 \varphi(\hat{x}) \exp\left[\frac{g}{2\pi} \int \bar{\varphi}(\hat{x}') S_R(\hat{x} - \hat{x}') P_2 \varphi(\hat{x}') \, \mathrm{d}^2 x'\right] + \\ + P_2 \varphi(\hat{x}) \exp\left[\frac{g}{2\pi} \int \bar{\varphi}(\hat{x}') S_R(\hat{x} - \hat{x}') P_1 \varphi(\hat{x}') \, \mathrm{d}^2 x'\right] .$$

3. – Connection with the Thirring solution.

The operators (11) are actually well-defined only in the Hilbert space in which the reference state $|0)$ is defined by

$$(17) \qquad \varphi_\tau(x)|0) = 0 , \qquad (0|0) = 1 \quad \text{for all } x, \ t \text{ being} = 0$$

and all the other states are defined by applying the « creation operators » $\varphi^*(x)$ any number of times to $|0)$. It is immediately clear that in this representation the operators $\varrho_\tau(x, t)$ are diagonal, their eigenvector being

$$\varphi_1^*(x_1) \dots \varphi_1^*(x_n)\varphi_2^*(y_1) \dots \varphi_2^*(y_{n'})|0) \,. \tag{18}$$

Consequently, the exponential operators and the operators ψ_τ have a well defined meaning.

Since $\varrho_\tau|0) = 0$ the reference state may be defined by

$$\psi_\tau(x, t)|0) = 0 \quad \text{(any fixed time)} \tag{19}$$

as well. This corresponds to the definition of THIRRING [1].

Let us now compute the energy-momentum vector by inserting the solution (11) into (4). We get:

$$P = \int_{-\infty}^{+\infty} p\left\{c_1^*(p)c_1(p) + c_2^*(p)c_2(p)\right\}\mathrm{d}p \,, \qquad E = \int_{-\infty}^{+\infty} p\left\{c_1^* c_1 - c_2^* c_2\right\}\mathrm{d}p \,, \tag{20}$$

P and E being the total momentum and energy of the interacting field. The eigenvectors of this two vectors are obviously

$$|k; p) = c_1^*(k_1) \dots c_1^*(k_n)c_2^*(p_1) \dots c_2^*(p_{n'})|0) = \tag{21}$$

$$= (2\pi)^{-(n+n')/2}\int \exp\Big[i\Big(\sum_\alpha k_\alpha x_\alpha + \sum_\beta p_\beta y_\beta\Big)\Big] \prod_{\alpha=1}^{n} \varphi_1^*(x_\alpha)\,\mathrm{d}x_\alpha \prod_{\beta=1}^{n'} \varphi_2^*(y_\beta)\,\mathrm{d}y_\beta|0) \,.$$

The corresponding energy eigenvalue $e = \sum_\alpha k_\alpha - \sum_\beta p_\beta$ is not positive definite. The reference state corresponds physically to a state where all the antiparticle states are occupied and all the particle states are empty [1].

To get the connection with the THIRRING approach we first remark that the Heisenberg operators ψ_τ^* taken at a definite time, say $t = 0$, can be alternatively used to construct the Hilbert space. The ψ's and the φ's are namely related by a unitary transformation:

$$\psi_\tau(x, t) = U^*(t)\varphi_\tau(x, t)\,U(t) \,, \qquad U^*(t)\,U(t) = 1 \,. \tag{22}$$

This unitary operator is obviously identical to the usual interaction representation operator $U(t, -\infty)$ and satisfies the well-known differential equation:

$$iU(t) = -\int \mathrm{d}x L_1(x, t)U(t) = g\int \overline{\varphi}(\overline{\varphi}\varphi)\varphi\,\mathrm{d}x U = \tag{23}$$

$$= -2g\int \varrho_1(x' - t)\varrho_2(x' + t)\,\mathrm{d}x' U(t) \,.$$

In deriving the last expression we have again used the fact that the square of a field operator vanishes because of the commutation relations. Since the density operators ϱ_τ commute for all the values of their arguments, we may treat (23) as a c-number differential equation, the solution being

$$U(t) = \exp\left[2ig\int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{+\infty} \mathrm{d}x' \varrho_1(x'-t')\varrho_2(x'+t')\right]. \tag{24}$$

It is very easy to check this formula directly by inserting it into (22).

Now, in order to express the state $|k;p)$ in terms of the ψ_τ^*-operators we make in (21) the substitution $\varphi_1^* = \psi_1^* \exp[ig\varrho_2]$, $\varphi_2^* = \psi_2^* \exp[ig\varrho_1]$ and commute all the exponential operators through to the right by using (13). The result of this straight-forward operation is

$$|k;p) = (2\pi)^{-(n+n')/2} \cdot \int \prod_{\alpha,\beta} \psi_1^*(x_\alpha)\psi_2^*(y_\beta)\,\mathrm{d}x_\alpha\,\mathrm{d}y_\beta\,\chi(x;y)\exp\left[i\left(\sum k_\alpha x_\alpha + \sum p_\beta y_\beta\right)\right]|0)\,, \tag{25}$$

$$\chi(x;y) = \exp\left[ig\sum_{\alpha,\beta}\theta_+(x_\alpha - y_\beta)\right] = \prod_{\alpha\beta}\left\{1 + (\exp[ig]-1)\theta_+(x_\alpha - y_\beta)\right\} =$$

$$= \prod_{\alpha,\beta} \exp[ig/2]\,\frac{1}{\sqrt{1+\lambda^2}}\left\{1 + i\lambda\varepsilon(x_\alpha - y_\beta)\right\},\quad \lambda = \operatorname{tg} g/2\,.$$

In deriving the last formula we have used the identity $2\theta_+(x) = 1 + \varepsilon(x)$ in order to make a comparison with the results of THIRRING. Apart from the insignificant phase-factor $\exp[ig/2]$ our formula disagrees with the corresponding formula of Thirring in as far as the constant λ in THIRRING's paper is equal to $g/2$ in our notation instead of $\operatorname{tg} g/2$. To explain the discrepancy we first note that, as shown by THIRRING, the function $\chi(x;y)$ satisfies the eigenvalue equation:

$$\begin{cases} e\chi = \left\{\sum_\alpha{}' \frac{1}{i}\frac{\partial}{\partial x_\alpha} - \sum_\beta{}' \frac{1}{i}\frac{\partial}{\partial y_\beta} - 2g\sum_{\alpha,\beta}\delta(x_\alpha - y_\beta)\right\}\chi\,, \\ e = \sum{}' k_\alpha - \sum{}' p_\beta\,, \end{cases} \tag{26}$$

this equation being the starting point of Thirring's method. Inserting here the function χ in its exponential form and performing the differentiation by means of the formula $\partial/\partial x \exp[f(x)] = \exp[f]\partial f/\partial x$ it is immediately seen that (26) is satisfied. On the other hand if *we define* $\varepsilon(x)\ \delta(x) = \varepsilon(0)\ \delta(x) = 0$ as THIRRING does, we get another solution:

$$\chi = \sum_{\alpha,\beta}\left\{1 + i\,\frac{g}{2}\,\varepsilon(x_\alpha - y_\beta)\right\}.$$

The explanation lies in the fact that the equation (26) must be defined as a limit of a non-singular equation, and this can be done in many different ways (*). If we replace $\delta(\xi)\chi(\xi)$ in (26) $D(\xi)\chi(\xi)$, where the non-singular function $D(\xi) \to \delta(\xi)$ we obtain in the limit our result (25). If however we replace $\delta(\xi)\ \chi(\xi)$, as Thirring does in his paper, by $D(\xi)\int D(\xi - \xi')\chi(\xi')\,\mathrm{d}\xi'$ the limit $D \to \delta$ will lead us to the result of Thirring. Now, the essential thing seems to be the fact that we always get in the limit a solution of the form $\prod_{\alpha,\beta}\{1+c\varepsilon(x_\alpha - y_\beta)\}$, where c may well turn out to be any function of the coupling constant if we adopt an appropriate limiting procedure $\int D(\xi, \xi')\cdot \cdot\chi(\xi')\,\mathrm{d}\xi'$ in order to define $\delta(\xi)\chi(\xi)$. The situation may be also understood by noting that instead of taking $\varepsilon(0) = 0$ we are allowed to define $\varepsilon(0)$ to be any complex number ($\varepsilon(0) = i(\mathrm{ctg}\ g/2 - 2/g)$, *e.g.* in order to get our result (25)). If we take the equation (26) as the starting point of the theory the above mentioned ambiguity will survive also when we pass to the physical representation.

The ambiguity becomes, however, harmless if we adopt the point of view that all the observable quantities of the theory should be expressed as functions of a fixed observable quantity rather than functions of the unobservable coupling constant ([6]).

4. – The S-matrix and the renormalization.

In order to make the solution (11) physically meaningful we must try to attach a mathematical meaning to all our operator relations in the representation with the true vacuum state $|0\rangle$ defined by $a(k)|0\rangle = b(p)|0\rangle = 0$, $\langle 0|0\rangle = 1$ (compare (9) and (10)). This amounts to express the exponential operators (11) in terms of ordered products in the sense of G. C. Wick ([7]). In the conventional treatment of perturbation theory one replaces ϱ_τ in the equation (23) for $U(t)$ by $:\varrho_\tau:$ where, according to Wick's convention, an operator between double points, means the corresponding ordered operator (all the annihilation operator a and b commuted to the right of the creation operators a^*, b^*, as if a, b, a^*, b^* were all anti-commuting). It is easily seen that the operators $:\varrho_\tau:$ still fulfill the commutation relations (12′), so that the formal solution (24) still holds after this replacement. Although $U(t)$ cannot have a well defined mathematical meaning according to Haag's theorem ([8]) (Haag's theorem on the non-existence of $U(0, -\infty)$ applies also in the 2-dimen-

(*) The following remarks I owe to Dr. Thirring, private communication.

([6]) G. Källén: *Nuovo Cimento*, **12**, 217 (1954).

([7]) G. C. Wick: *Phys. Rev.*, **80**, 268 (1950).

([8]) R. Haag: *Dan. Mat. Fys. Medd.*, **29**, 12 (1955).

sional case), we get for $t \to +\infty$ a perfectly sound S-matrix by this procedure:

$$(27)\qquad \begin{cases} S = \exp\left[igQ_1Q_2\right] \equiv U(+\infty, -\infty)\,, \\ Q_{1,2} = \int_{-\infty}^{+\infty} :\varphi_2^1\varphi_2^1(y): \mathrm{d}y = \int_{-\infty}^{+\infty} \theta_{\pm}(p)\left\{a^*(p)a(p) - b^*(p)b(p)\right\}\mathrm{d}p\,. \end{cases}$$

The S-matrix is obviously unitary. It is, however, diagonal in the physical particle representation:

$$(28)\qquad S\,|k_1,\ldots,k_n;\ p_1,\ldots,p_{n'}\rangle =$$
$$= \exp\left[ig\left[\sum_{\alpha}\theta_+(k_\alpha) - \sum_{\beta}\theta_+(p_\beta)\right]\left[\sum_{\alpha}\theta_-(k_\alpha) - \sum_{\beta}\theta_-(p_\beta)\right]\right]|k_1,\ldots,k_n;\ p_1,\ldots,p_{n'}\rangle\,.$$

There is no creation of particles and the elastic scattering reduces to a simple penetration of particles with only a relative change of phase of the plane waves. This fact is connected with the existence of the constants of motion Q_1 and Q_2 (*).

However, in order to be consistent, let us investigate the meaning of the solution (11) in the physical representation. As we have already pointed out, this amounts to express the exponential operators $\exp\left[igQ_1(v)\right]$, $\exp\left[igQ_2(u)\right]$ in terms of ordered products. We shall study only the case of $\exp\left[ig'Q_2(u)\right] \equiv e_2(u)$, the corresponding expression for $e_1(v)$ being obtainable from $e_2(u)$ by a simple space-reflection.

We start from the remark that $e_2(u)$ obeys the differential equation

$$(29)\qquad \frac{1}{i}\frac{\partial}{\partial g}e_2(u) = Q_2(u)e_2(u) \qquad \text{with } e_2 = 1,\ \text{for } g = 0\,.$$

We shall try to push the annihilation operators contained in $Q_2(u)$ to the right of e_2 keeping the creation operators on the left-hand side of e_2. An expression where this has already been done, we shall enclose in semi-colons: $;Q_2(u)e_2(u);$ (the semi-colons should denote the operation: commute the annihilation operators to the right of e_2 as if all the field operators contained in Q_2 were commuting with e_2 and anticommuting among themselves). The commutation relations of e_2 with $a(p)$ and $b(p)$ we shall get with the help of the formula (13), which reads in momentum representation:

$$(30)\qquad e^{-1}(u)C(p)e(u) = C(p) + \frac{1 - \exp\left[ig\right]}{2\pi i}\int_{-\infty}^{+\infty}\frac{\mathrm{d}p'}{p_+ - p'}\,C(p') \equiv (1 + K)C\,,$$

where $C(p) = c_2(p)\,\exp\left[ipu\right]$.

(*) Compare, W. E. Thirring (2).

Here we have dropped the subscript 2 and introduced the abbreviation K for the integral operator. $p_\pm$ means $p \mp i\varepsilon$ with $\varepsilon \to +0$. Using the notation

$$C_+(p) = \theta_+(p) b^*(-p) \exp[ipu], \quad C_-(p) = \theta_-(p) a(p) \exp[ipu], \quad C_+ + C_- = C,$$

we can decompose the equation $Ce = eC + KeC$ which follows from (30) into its creation and annihilation parts:

$$\begin{cases} C_+ e = eC_+ + K_+ eC_+ + K_- eC_- & (p > 0), \\ C_- e = eC_- + K_+ eC_+ + K_- eC_- & (p < 0). \end{cases} \tag{31}$$

$K_\pm$ are a shorthand notation for the integral operators

$$K_\pm f(p) \equiv \frac{1 - \exp[ig]}{2\pi i} \int_{-\infty}^{+\infty} \frac{dp'}{p_\pm - p'} \theta_\pm(p') f(p'). \tag{32}$$

Formula (31) enables us to express eC_+ and $C_- e$, which have the creation and annihilation operators on the « wrong » side of e, in terms of $C_+ e$ and eC_-. From the first equation (31) we namely get

$$eC_+ = (1 + K_+)^{-1}(C_+ e - K_- eC_-). \tag{33}$$

We insert this into the second equation (31) with the result

$$\begin{aligned} C_- e &= (1 + K_-) eC_- + K_+ (1 + K_+)^{-1}(C_+ e - K_- eC_-) = \\ &= \{1 + (1 + K_+)^{-1} K_-\} eC_- + \{1 - (1 + K_+)^{-1}\} C_+ e. \end{aligned} \tag{34}$$

Here we have tacitly assumed that the integral equation $(1 + K_+) f = g$ can be uniquely solved for g. Indeed, a very similar equation was encountered by Thirring in the computation of the matrix element $\langle 3 | \psi | 0 \rangle$, the unique solution being:

$$\begin{cases} (1 + K_+)^{-1} = 1 + G_+, G_+(p, p') = G(p, p') \theta_+(p'), \\ G(p, p') = \dfrac{1 - \exp[-ig]}{2\pi i} \left(\dfrac{p_+}{p'_-}\right)^{g'/2\pi} \dfrac{1}{p_+ - p'}, \\ g' = g + 2n\pi, \ n = 0, \pm 1, \pm 2, \ldots, \end{cases} \tag{35}$$

(see Appendix). Here G_+ is actually defined only for $p, p' > 0$, but it will turn out to be very convenient to prolong it analytically for all real values of both p and p' as indicated by the expression G:

$$p_+^{g'/2\pi} = |p|^{g'/2\pi} \text{ for } p > 0 \text{ and } p'^{-g'/2\pi}_- = |p'|^{-g'/2\pi} \text{ for } p' > 0 ;$$

p_+ has to be continued to negative values over the positive imaginary axis, p'_- over the negative imaginary axis. The integer n is to be chosen so as to keep $g'/2\pi$ always in the internal $-\frac{1}{2} < g'/2\pi \leqslant +\frac{1}{2}$. Furthermore we have

$$(1+K_+)^{-1} K \equiv (1+G_+) K = -G \tag{36}$$

for *all* values of p and p'. (This relation is certainly true for $p > 0$, $p' > 0$, that simply follows from the definition of the resolvent G. By analytical continuation (see (31)) it must hold for all p and p'). Using (35) and (36) a simple algebra will lead us to the following compact formula for (33) and (34):

$$[e, C(p)] = G; C(p)e; \equiv \frac{1 - \exp[-ig]}{2\pi i} \int_{-\infty}^{+\infty} \frac{dp'}{p_+ - p'_-} \left(\frac{p_-}{p'_-}\right)^{g'/2\pi}; C(p')e; \tag{37}$$

which holds for all values of p. Taking the hermitian conjugate of (37) and changing g into $-g$ we get also

$$[C^*(p), e] = G_T; C^*(p)e ; \quad \text{with} \quad G_T(p, p') = G(p', p) . \tag{38}$$

Returning to the differential equation (29) we first transcribe it into momentum space:

$$\frac{1}{i}\frac{\partial}{\partial g} e(u) = \frac{1}{2\pi i} \int \frac{dp\, dp'}{p'_- - p^+} C^*(p) C(p') e(u) . \tag{39}$$

It is clear now, that the expression C^*Ce in (39) can be rearranged by the help of (37) and (38) into a « semi-ordered product », so that (39) is certainly reducible to the following form

$$\frac{1}{i}\frac{\partial}{\partial g} e(u) = \int dp\, dp' A_g(p, p') ; C^*(p) C(p') e(u) ; + B_g e(u) , \tag{40}$$

where A_g is a certain c-number kernel depending also on g and B_g is a c-number function of g only (the constant B can in principle arise from the commutator of C^* and C in the process of ordering). On the other hand it is easy to verify

the general formula:

$$\frac{\partial}{\partial\lambda} : \exp\left[Q_\lambda(C^*, C)\right] : = : Q'_\lambda e^{Q_\lambda} : = ; Q'_\lambda : e^{Q_\lambda} : ;$$

where Q_λ is any quadratic function of C^*, C depending on the parameter λ. This formula enables us to solve the equation (40) in the form of an exponential operator ordered in the sense of Wick:

$$(41) \qquad e(u) = : \exp i\int_0^\infty \mathrm{d}g \left\{ \int \mathrm{d}p\, \mathrm{d}p' A_g C^* C + B_g \right\} : =$$

$$= Z^{\frac{1}{2}} : \exp\left[\int H(p, p') C^*(p) C(p') \cdot \mathrm{d}p\, \mathrm{d}p' \right] : ,$$

where

$$H = i\int \mathrm{d}g A_g , \qquad Z^{\frac{1}{2}} = \exp\left[i\int \mathrm{d}g B_g \right].$$

We are left now only with the task of computing $Z^{\frac{1}{2}}$ and H. Instead of following the procedure outlined above, it is much easier—knowing once the general form of the solution—to start from the simple remark that

$$(42) \qquad \langle 0 \,|\, e(u) \,|\, 0 \rangle = Z^{\frac{1}{2}} , \qquad \langle 0 \,|\, \{ C^*(p'), [C(p), e(u)] \} \,|\, 0 \rangle = Z^{\frac{1}{2}} H(p, p') .$$

Inserting (37) for the commutator $[C, e]$ into the second expression (42), the vacuum expectation value can be immediately calculated with the result

$$(43) \qquad H(p, p') = - G(p, p')$$

(the factor $Z^{\frac{1}{2}}$ drops out). In order to obtain $Z^{\frac{1}{2}}$ we take the vacuum expectation value of (39) and insert in the right-hand side ofr $e(u)$ the expression (41). After a straightforward evaluation of the matrix element on the right-hand side we get

$$(44) \qquad \frac{1}{i}\frac{\partial}{\partial g}\langle e \rangle = -\frac{1}{2\pi i}\int \frac{\mathrm{d}p\, \mathrm{d}p'}{p_+ - p'} \{ \theta_+(p)\delta(p - p') +$$

$$+ \theta_+(p) H(p', p) \theta_-(p') \} \langle e \rangle \equiv I \langle e \rangle , \qquad \langle e \rangle = Z^{\frac{1}{2}} = \exp\left[i\int_0^g I \,\mathrm{d}g \right].$$

The integral I is divergent.

Collecting all the results of this section we finally arrive at the following

formula for ψ_1:

$$(45)\quad \psi_1(x,t) = \varphi_1(v) Z^{\frac{1}{2}} : \exp\left[\frac{\exp[-ig]-1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{p_+}{p'_-}\right)^{g'/\pi}\frac{\exp[-i(p-p')u]}{p_+ - p'_-}c_2^*(p)\cdot\right.$$
$$\left.\cdot c_2(p')\,\mathrm{d}p\,\mathrm{d}p'\right] : .$$

The corresponding expression for ψ_2 we get by space-refraction

$$i\psi_2(x,t) = R^{-1}\psi_1(-x,t)R, \qquad R^{-1}c_2^*(-p)R = ic_1(p),$$
$$R = \exp\left[\frac{i\pi}{2}\int \mathrm{d}p\,\{c_1^*(p)c_2(-p) + \text{c.c.}\}\right],$$

$$(46)\quad \psi_2(x,t) = \varphi_2(u) Z^{\frac{1}{2}} : \exp\left[\frac{\exp[-ig]-1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{p_+}{p'_-}\right)^{g'/2\pi}\frac{\exp[i(p-p')v]}{p_+ - p'_-}\cdot\right.$$
$$\left.\cdot c_1^*(-p)c_1(-p')\,\mathrm{d}p\,\mathrm{d}p'\right] : .$$

We immediately see from (45) and (46) that

$$\langle 0\,|\,\psi_r(x,t)\,|\,k\rangle = Z^{\frac{1}{2}}\langle 0\,|\,\varphi_r(x,t)\,|\,k\rangle\,,$$

where $|k\rangle$ is a one-particle state. As we have anticipated with our notation $Z^{\frac{1}{2}}$ is an infinite field-renormalization constant. The renormalized field-operators defined by $\psi_{rr} = Z^{-\frac{1}{2}}\psi_r$ have obviously finite matrix elements in the physical representation. For the matrix element $\langle 3\,|\,\psi_{r1}\,|\,0\rangle$, *e.g.*, formula (45) gives:

$$(47)\quad \langle p_1 p_2 k\,|\,\psi_{r1}(x,t)\,|\,0\rangle = -\langle p_1\,|\,\varphi_1(v)\,|\,0\rangle\cdot$$
$$\cdot\frac{\sin g/2}{\pi}\left|\frac{k}{p_2}\right|^{g'/2\pi}\frac{\exp[-i(k+p_2)u]}{k+p_2}\cdot\theta_+(p_1)\theta_-(p_2)\theta_-(k)\,,\quad (p_1>0,\ p_2>0),$$

($p_{1,2}$ are anti-particles, k a particle). This is an analytic function of the coupling constant g for $|g|\leqslant\pi$ because in this region $g' = g$. The same statement is obviously true for all matrix elements. With the substitution $\lambda = \operatorname{tg} g/2$ (47) agrees with the corresponding formula of Thirring (see the discussion at the end of the foregoing paragraph).

In order to check the correctness of the expression (27) for the S-matrix let us compute the asymptotic limit of the renormalized field-operators for $t\to+\infty$. By means of the well known formula

$$\lim_{t\to+\infty}\frac{\exp[-ixt]}{x_+} = -i\lim\int_{-\infty}^{t}\exp[-ixt]\,\mathrm{d}t = -2\pi i\,\delta(x)\,,$$

we get from (45) and (46):

$$\psi_{\text{out}1,2}(x,t) = \varphi_{1,2}(x,t) : \exp\int \mathrm{d}p\, \theta_{\pm}(p)\{\exp[ig]-1)a^*(p)a(p) + \\ + (\exp[-ig]-1)b^*(p)b(p)\} : , \tag{48}$$

(note that $\exp[ig'] = \exp[ig]$). Using the identity

$$\exp\left[\int f_\lambda(p)a^*a(p)\,\mathrm{d}p\right] = : \exp\left[\int \mathrm{d}p \int_0^\lambda \mathrm{d}\lambda \frac{\partial}{\partial\lambda} f_\lambda(p)a^*a(p)\right] : ,$$

where f_λ is any function of λ and p, (48) can be cast into the form of an ordinary exponential operator:

$$\psi_{\text{out}1,2}(x,t) = \varphi_{1,2}(x,t)\exp[igQ_{1,2}]\,, \tag{49}$$

where $Q_{1,2}$ is defined by (27). This form makes it immediately clear that

$$\psi_{\text{out}\,\tau} = S^{-1}\varphi_\tau S \qquad \text{with} \qquad S = \exp[igQ_1Q_2], \tag{50}$$

which confirms our previous result (*).

Let us mention that we could also find an explicit expression for the operator products $\psi_\tau(x,t)\psi_\tau^*(x',t)$ and $\psi_\tau\psi_\tau'$, as functionals of the incoming field by using exactly the same method. According to the formulae (14) to (14‴) this amounts only to the ordering of the operator

$$\exp\left[ig\int_{u'}^{u} \varrho_2(u)\,\mathrm{d}u\right].$$

This problem, as well as the problem of infrared divergencies, will, however, not be dealt with in this paper.

* * *

In conclusion, the author would like to thank Professor B. Ferretti and Dr. A. Petermann for their interest in this work and for many illuminating discussions. He is especially indebted to dr. W. E. Thirring for detailed discussions on many vital points concerning this paper.

(*) From (49) the S-matrix can be determined only up to an irrelevant phase-factor.

Appendix

In order to solve the integral equation $(1+K_+)f=g$, or explicitly:

$$f(p)+\frac{1-\exp[ig]}{2\pi i}\int_0^\infty \frac{f(p')}{p_+-p'}\,dp'=g(p)\,,\quad p>0\,. \tag{A.1}$$

we write it in the form

$$h(p)+\frac{1-\exp[ig]}{2\pi i}\int_0^\infty \frac{h(p')+g(p')}{p^+-p'}\,dp'=0\,, \tag{A.2}$$

where $h(p)=f(p)-g(p)$.

From here we conclude that $h(p)$ is the limiting value $h(p_+)$ of a function analytic in the whole complex p-plane with the exception of a cut along the positive real axis. The « jump relation » gives

$$h(p_+)-h(p_-)-(1-\exp[ig])[h(p_+)+g(p)]=0,\quad h(p_+)\equiv h(p),\quad p>0. \tag{A.3}$$

By introducing a new function

$$p^{g'/2\pi}S(p)=h(p)\,,$$

where $g'=g+2n\pi$, $n=0,\ \pm1,\ \pm2,\ \ldots$ (A.3) becomes

$$S(p_+)-S(p_-)=-2\pi i\,\frac{1-\exp[-ig]}{2\pi i}\,p_+^{-g'/2\pi}g(p)\,, \tag{A.4}$$

because of $p_-^{g'/2\pi}=\exp[ig']p_+^{g'/2\pi}=\exp[ig]p_+^{g'/2\pi}$.

The solution of (A.4) is given by the Cauchy formula

$$S(p)=\frac{1-\exp[-ig]}{2\pi i}\int_0^\infty p_+'^{-g'/2\pi}\,\frac{g(p')}{p-p'}\,dp',\qquad p \text{ complex}. \tag{A.5}$$

Going back to $f(p)$ we arrive at the solution:

$$f=(1+G_+)g\,,\qquad G_+(p,p')=\frac{1-\exp[-ig]}{2\pi i}\left(\frac{p_+}{p'_+}\right)^{g'/.\,\pi}\frac{1}{p_+-p'}\,,\quad p,p'>0\,, \tag{A.6}$$

of the integral equation (A.1). Here we may take without loss of generality the branch $p_+^{\pm g'/2\pi} = |p|^{\pm g'/2\pi}$ for $p > 0$. To get the definition used in formula (35) of the text we write $p_-'^{-g'/2\pi} = |p'|^{-g'/2\pi}$ for $p' > 0$ instead of $p_+'^{-g'/2\pi}$, the-sign-meaning simply a prescription how to get away from the cut in the case of an eventual analytical prolongation in the variable p'.

The value of the arbitrary integer n, which we have treated rather formally so far, has to be restricted by the condition that the integrals $K_- G_+$ and $G_+ K_+$ be convergent for $p = 0$ and $p = \infty$. This condition gives $|g'/2\pi| < 1$. It can be always fulfilled choosing n so as to have *e.g.* $-\frac{1}{2} < g'/2\pi \leqslant \frac{1}{2}$.

RIASSUNTO (*)

Si risolve il modello bidimensionale di una teoria relativistica recentemente proposta da W. E. Thirring (1) sviluppando l'operatore di campo ψ come funzionale esplicito del corrispondente campo entrante. Si trova che la matrice S (senza sorgenti esterne) è della forma $S = \exp[iQ_1 Q_2]$, dove $Q_{1,2}$ sono due costanti del moto. La matrice S è unitaria, ma dà luogo solo ad un cambiamento relativo di fase delle onde piane associate alle particelle che collidono, tutte le sezioni d'urto essendo uguali a zero (2). Dopo la rinormalizzazione tutti gli elementi di matrice dell'operatore di campo risultano funzioni analitiche, finite della costante di accoppiamento. Si discute un'apparente discrepanza coi risultati di Thirring (1).

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

On Interacting Spinor Fields in One Dimension.

W. Thirring

University of Washington - Seattle, Washington (*)

(ricevuto il 23 Maggio 1958)

Summary. — The physical significance of the exact solution of interacting spinor fields in one space dimension is discussed. It is argued that the absence of ghosts in this theory is connected with the fact that there is only a field but no infinite charge renormalization.

1. – Introduction.

In order to study in detail the alleged inconsistencies of relativistic field theories, the author investigated some time ago a selfcoupled mass-less two-dimensional spinor field (1). Although this theory proved to be soluble, it did not serve its original purpose since it turned out to be consistent. The theory had all the properties one would like to see in a renormalizable theory and did not misbehave in any major aspect. Subsequently (2) V. Glaser, by a brilliant method, succeeded in exhibiting explicitely all matrix elements of the field operators between physical states. Since both of these papers on the subject concentrate on the mathematical and formal aspects of the problem, we will discuss in this note the physical significance of our findings. In particular, one might wonder which features of a three-dimensional theory are included in this model and which are not, in order to decide whether this result has any bearing on the problem of three-dimensional renormalizable theories. This shall be done in the two subsequent sections. First, we shall quote the

(*) Now at Physikalisches Institut, Universität Bern and CERN, Geneva.

(1) W. Thirring: *Ann. Phys.*, **9**, 91 (1958), Henceforth quoted as I.

(2) *An explicit solution of the Thirring model.* Proceeding paper. Henceforth quoted as II.

important mathematical properties of the model and compare them with results of fashionable approximation methods in field theory.

The problem at hand is characterized [3] by the usual interaction term

$$L' = g\bar{\psi}\psi\bar{\psi}\psi \tag{1}$$

and we shall summarize the Glaser solution by expressing ψ in terms of the incoming field φ:

$$\psi(x) = \left\{ P_1 \exp\left[2g \int \mathrm{d}^2x' : \bar{\varphi}(x') S_r(x - x') P_2 \varphi(x') : \right] + \right.$$
$$\left. + P_2 \exp\left[2g \int \mathrm{d}^2x' : \bar{\varphi}(x') S_r(x - x') P_1 \varphi(x') : \right] \right\} \varphi(x) , \tag{2}$$

with $P_{\frac{1}{2}} = \frac{1}{2}(1 \pm \gamma_5)$.

Here $\varphi(x)$ is a solution of the free wave equation, obeying the usual commutation rules. One can check that (1) actually satisfies the appropriate field equations and canonical commutation rules. The physical vacuum $|0\rangle$ is then the state which is annihilated by the positive frequency part of φ and the other physical states are obtained by applying the negative frequency part of φ unto $|0\rangle$. For exhibiting the matrix elements of ψ between these states, one has to order the exponentials in (1). This can be done with the result

$$\exp 2g \int \mathrm{d}^2x' : \bar{\varphi}(x') S_r(x - x') P_1 \varphi(x') : = \tag{3}$$

$$= Z^{\frac{1}{2}} : \exp\left[\frac{\exp[-ig] - 1}{2\pi i} \int_{-\infty}^{\infty} \mathrm{d}p \, \mathrm{d}p' \varphi^+(p) \left(\frac{p_+}{p_-} \right)^{g'/\pi} \frac{\exp[i(x_0 + x_1)(p' - p)]}{p_+ - p_-} P_1 \varphi(p') \right] : ,$$

where

$$p_\pm = p \pm i\varepsilon , \qquad \varphi(p) = \int \mathrm{d}x_1 \exp[-ipx_1] \varphi(x_1, 0) ,$$

and

$$g' = g + 2\pi n \quad \text{such that} \quad |g'/2\pi| < 1 .$$

Here Z is the field renormalization constant, a quantity which tends to zero and can also be calculated explicitly

$$Z = \exp - \int_0^g \mathrm{d}f \frac{\sin(f/2)}{\pi^2} \int_0^\infty \frac{\mathrm{d}p \, \mathrm{d}p'}{(p + p')^2} \left(\frac{p}{p'} \right)^{f/\pi} . \tag{4}$$

---

[3] For notations and formal details see I.

All the matrix-elements of the renormalized field $\psi_r = \psi Z^{-\frac{1}{2}}$ are finite for finite momenta. For instance, for $\langle 3 | \psi_{1r} | 0 \rangle$ we find

$$(5) \qquad \langle 3 | \psi_{1r}(x,t) | 0 \rangle = \langle 1 | \varphi_1(x,t) | 0 \rangle \frac{\sin(g/2)}{\pi} \cdot$$

$$\cdot \langle 2 | \int_0^\infty \mathrm{d}p \, \mathrm{d}p' \left(\frac{p}{p'}\right)^{g'/2\pi} \frac{\exp\left[i(x_0 + x_1)(p' + p)\right]}{p + p'} \varphi_2^+(p) \varphi_2(p) | 0 \rangle \, .$$

It is interesting to note that this exact result agrees with an expression which was obtained [4] by the renormalization group technique. This shows that the renormalization group method works in this case. Another method which is based on a summation of a subset of diagrams is the ladder approximation to the Bethe-Salpeter equation. With the aid of (3) one can evaluate the exact two-body wave function which will be done in Sect. **3**. The two body equation in the ladder approximation has many solutions [5], only few of which show any resemblance to the exact result.

The expression (3) differs from the one obtained by a more old-fashioned method (I) inasmuch as in the latter g is replaced by $\operatorname{arctg} g$. This discrepancy arises from the fact that the strength of a δ-interaction for Dirac particles depends on the limiting procedure by which it is evaluated. Since the interaction of the theory corresponds to scattering of particles by antiparticles via a δ-function, the problem was attacked in (I) by solving the corresponding many-body Dirac equation. Now the solution of an equation

$$(6) \qquad \frac{\partial}{\partial x} U(x) = ig \, \delta(x) U(x) \, ,$$

is not unique. Depending on whether one interprets $\delta(x)U(x)$ as

$$(7) \qquad \lim_{D \to \delta} D(x) U(x) \, , \qquad \text{or} \qquad \lim_{D \to \delta} D(x) \int \mathrm{d}' x D(x') U(x') \, ,$$

for instance, one gets

$$(8) \qquad U(x) = \exp\left[ig\theta(x)\right] \simeq 1 + i \operatorname{tg}(g/2) \cdot \varepsilon(x) \qquad \text{or} \qquad U(x) = 1 + i(g/2) \cdot \varepsilon(x) \, .$$

The solution in (I) corresponds to adopting the latter and the one of (II) to the former limiting procedure. The solution with $\exp\left[ig\theta(x)\right]$ shows a funny

---

[4] M. E. Mayer and D. V. Shirkov, to be published in *Dokl. Akad. Nauk USSR*. The author is indebted to Drs. Mayer and Shirkov for informing him of their results. Earlier calculations by those authors have been reported in a preliminary paper, but have been withdrawn since.

[5] F. Scarf: to be published.

periodicity according to which a coupling constant $g = 2\pi$ is as good as having no coupling at all. (This is characteristic for the Glaser solution, as can be seen from (2) and (3).) In the second form of the solution, the periodicity interval $-\pi$, π corresponds to $-\infty < g < \infty$ which seems more reasonable and was, therefore, adopted in (I). On the other hand, integrating the equations in the Heisenberg representation leads automatically to the periodic form of the solution. Those ambiguities in the solutions are well-known in field (6) theory—they correspond to the finite renormalizations in electrodynamic calculations. The form of the limiting process comes in when one takes differences between infinite expressions and this occurs also in our model, although there is only a diverging field but no charge renormalization. However, since only the factor of $\varepsilon(x)$ in (8) occurs in observables, there is no ambiguity in the relations between observable quantities, but only their relation to the g in (1) is arbitrary.

The matrix element (4) is significant for the propagation function inasmuch as $\langle 3 | \psi_r | 0 \rangle$ contributes to the mass-spectrum $r(c)$ of the propagation function (see I). This quantity has been calculated in I to second order in g to

$$r(c) = \left(\frac{g}{2\pi}\right)^2 \frac{1}{|c|},$$

and it had been mentioned that the exact contribution from $\langle 3 | \psi_r | 0 \rangle$ is the same function of c multiplied by a function of g. Indeed, one easily finds that this function is

$$\frac{4}{g^2} \sin^2 (g/2) \int_0^1 \mathrm{d}x \, \mathrm{d}y \, \delta(x+y-1) \left(\frac{x}{y}\right)^{g/\pi} = \frac{8}{g^2} \sin^2 (g/2) \frac{\Gamma(1-g/\pi)\Gamma(1+g/\pi)}{\Gamma(3)} = \frac{2}{g} \operatorname{tg} (g/2),$$

so that the contributions of one and three particle states to $r(c)$ give

$$r(c) = \delta(c) + \frac{g \operatorname{tg} (g/2)}{2\pi^2} |c|^{-1}.$$

This result is instructive inasmuch as it had been conjectured (7) that in the high energy limit $r(c)$ approaches its Born approximation value calculated with the unrenormalized coupling constant. Since in our model there is no mass, every finite energy corresponds to the high energy limit. According to what had been said before, this conjecture is not well defined in our case since

(6) See, for instance, KARPLUS and KROLL: *Phys. Rev.*, **77**, 536 (1950).
(7) A. KÄLLÉN: *CERN Symposium*, 1956.

it is not clear as to whether g or tg g is to be considered as the unrenormalized coupling constant. It actually turns out that it is the Born approximation taken neither with g nor with tg g but with the geometric mean of both.

2. – Elementary processes and one particle states.

Wave packets of massless particles in one dimension, like in three dimensions, do not diffuse; they are of the form $f(t \pm x)$. The latter form suggests that one can talk invariantly about particles going to the left and to the right. Indeed, one finds the fields $\psi_1 = P_1\psi$ and $\psi_2 = P_2\psi$ create positive (or destroy negative) particles going to the left and to the right respectively. This split of the field is obviously invariant under the proper Lorentz-group and is similar to the split in left-handed and right-handed particles in three dimensions. Mathematically speaking, the one-dimensional Lorentz group is abelian and has only one-dimensional irreducible representations. Adding reflections one has two-dimensional representations which, however, represent the proper subgroup reducibly. Furthermore, the charges of the particles going to the left and to the right, $Q_{1,2}$, are conserved separately, corresponding to a two parameter invariance group $\psi \to \exp[i(\alpha+\beta\gamma_5)]\psi$ of the theory. It turns out that in a local theory those conservation laws are guaranteed by the exclusion principle which requires $\psi_1^2 = \psi_2^2 = 0$ and, therefore, eliminates many point interactions. We shall now study the elementary processes between bare particles and discuss later how they can develop into real processes.

According to the above remark, it is obvious that $(1) = 2g\psi_1\psi_2\psi_1^T\psi_2^T$ is the most general local interaction term. Since each ψ consists of a creation and a destruction operator (1) contains many elementary processes. We shall discuss them by schematizing the directions of the incident and outgoing particles by arrows and also indicate their charge. (1) contains a term which creates four particles out of the vacuum. This process must be of the form  , other possibilities like  being forbidden.

Consequently the physical vacuum is not the bare vacuum but corresponds to pairs of bare particles going to the left and to the right. Since this state is to be an invariant state, the momentum distribution of the pairs must be continuous and this state cannot be analyzed into states with a finite number of bare particles. (Haag's theorem). The elementary process which makes three particles out of one is  and its mirror and charge

images.  and  are forbidden by the exclusion principle.

As possible transitions between two particle states we have  ,

whereas  ,  ,  ,  are forbidden.

The latter being only excluded by momentum conservation.

We shall now investigate how those processes in succession build up the physical particle state. We have seen that a bare left particle will emit pairs to the right and one would expect that each of these right particles starts emitting pairs to the left. This, however, does not happen, since the right particles, coming from the same point and going with the same speed in the same direction, neutralize each other completely. The source for particles to the left is proportional to the local charge going to the right and we get indeed $\psi_2^+(x)\psi_2(x) = \varphi_2^+(x)\varphi_2(x)$. That is to say, two bare right particles at the same point and opposite charge are already two physical particles going to the right, no further virtual processes will occur. Correspondingly the pairs to the right will undergo no further processes and a bare particle to the left will develop into a physical particle to the left and pairs of physical particles to the right. This statement is mathematically expressed by

$$\psi_1(x,t)\,|0\rangle = Z^{\frac{1}{2}}\varphi_1(x,t)\cdot$$

$$\cdot\exp\left[\frac{-\sin(g/2)}{\pi}\int_0^\infty \mathrm{d}p\,\mathrm{d}p'\varphi_2^+(p)\left(\frac{p}{p'}\right)^{g'/\pi}\frac{\exp[i(t-x)(p+p')]}{p+p'}\varphi_2(p')\right]|0\rangle\,,$$

which follows from (3). Although this expression contains the diverging quantity Z the physically significant relative probabilities are finite as discussed in (I). Similar effects happen in the temporal development of a state with two bare particles. This can be discussed along the same lines but we shall not dwell on those trivial points but work out the exact two body wave function in the next section.

3. – Two particle states and renormalization.

In this section we shall derive formally the two body wave function and afterwards discuss its significance for renormalization in this theory. The two body wave function $X(x, x')$ is defined as the probability amplitude for finding

bare particles at the space-time points x and x' in a state of two physical particles. With the aid of (2) this can be readily worked out for two positive particles with momentum $p > 0$ and $q < 0$.

We find

$$X(x', x'') = \langle 0 \,|\, \psi_1(x')\psi_2(x'') \,|\, 2 \rangle = \frac{\exp\,[i(qx' + px'')]}{2\pi}\, \Phi(x' - x'')\,,$$

$$\varphi(x) = (\theta(r) + \exp\,[ig]\,\theta(-r)) \left(1 + \frac{\sin\,(g/2)}{\pi} \int_{|q|}^{\infty} \frac{\mathrm{d}q'}{q'} \exp\,[iq'c] \left(\frac{q'}{|q'|} - 1\right)^{g'/2\pi}\right) \cdot$$

$$\cdot \left(1 + \frac{\sin\,(g/2)}{\pi} \int_{p}^{\infty} \frac{\mathrm{d}p'}{p'} \exp\,[ip'u] \left(\frac{p'}{p} - 1\right)^{g'/\pi}\right) \cdot$$

$$v = (X_0 + X_1)\,, \qquad U = (X_0 + X_1)\,.$$

We have written the terms in such a way that for $u \to \infty$, and $v \to \infty$ the two brackets approach one. This shows that for $t = t'$, $u = -v$ the wave function differs for $u \to \infty$ from the one for $u \to -\infty$ by a phase shift $\exp\,[ig]$. For small separations the wave function has a rather complicated behaviour. This is to be compared with the wave function in the ladder approximation, which ([5]) can be expressed by hypergeometric functions and is not even periodic in g.

Our value for the phase shift agrees with the general result of Glaser that $S = \exp\,[igQ_1Q_2]$. This S-matrix has eigenvalue 1 for zero and one-particle states and generally for all states where all particles go in the same direction. This is intuitively expected since all particles have the same speed so that nothing can happen. S, however, is diagonal for eigenstates of Q_1 and Q_2. To get transitions one has to consider particles whose charge is not well defined. For instance, if we consider the states

$$|\pm\rangle = \frac{1}{\sqrt{2}} \left(|\overset{+}{\rightarrow}\overset{-}{\leftarrow}\rangle \pm |\overset{+}{\rightarrow}\overset{+}{\leftarrow}\rangle \right) \qquad \text{we find} \qquad S\,|+\rangle = \cos g\,|+\rangle + i \sin g\,|-\rangle\,,$$

so that $\sin^2 g$ is the probability for this kind of exchange reaction. The situation is similar to the scattering by a spherical potential where S is diagonal for the eigenstates of the angular momentum. In this case the initial condition of a plane wave requires mixtures of those eigenstates whereas in our model with its poor geometry this necessity does not arise. Nevertheless, the states $|\pm\rangle$ should be as good as eigenstates of $Q_{1,2}$ and one can say that g is directly related to scattering processes. This point will be important for the following discussion of renormalization.

The essential feature of the above result is that after dividing the factor Z the wave function exists; the ratio between incoming wave and scattered wave being a certain expression different from zero. This means that the probability for finding two bare particles in a state of two physical particles goes to zero like the square of the probability of finding one bare particle in a one-particle state. However, dividing by the latter factor we arrive at a non-zero expression and furthermore the probability for the particles being scattered is also finite. In fact, the ratio between the amplitudes for scattering and non-scattering is given by the same expression as the ratio of the amplitudes for finding one and three physical particles in a bare particle, which we found to be finite. This is the main difference to other models where the theory without cut-off exists, but no interaction occurs. In the Lee-model ([8]), for instance, the two-body wave function is finite after renormalization, however, baring non-hermitian interactions, the scattering amplitude goes to zero.

For discussing why this difficulty does not occur in our model, we have to look for an expression which defines the interaction strength in our theory. A characteristic expression ([9]) for the effective interaction strength for a scalar field is $\langle A\,|\,(\Box - m^2)\Phi_r\,|\,B\rangle$, where A and B are physical one-particle states and Φ_r is the renormalized field operator. This matrix element dominates the low energy scattering and is directly related to observable quantities. In the Lee-model this expression is g_0 times the probability amplitude for finding a bare V in the physical V state and this product goes to zero even if $g_0 \to \infty$. In our case the effective interaction is suitably defined by the matrix element of $\partial\psi_r$ between a one and a two-particle state:

$$\frac{g}{\sqrt{Z}}\langle 1\,|\,\psi_1\psi_2^+\,\psi_2\,|\,2\rangle\,.$$

This matrix element is, as explained below, finite, since the renormalization constants cancel out because of charge conservation. To see this we note that $\langle 1\,|\,\psi_r\,|\,0\rangle$ is finite and

$$\langle 1\,|\,\psi^+(x)\psi(x)\,|\,1\rangle = \langle 1\,|\,\varphi^+(x)\varphi(x)\,|\,1\rangle = Z\langle 1\,|\,\psi_r^+(x)\psi_r(x)\,|\,1\rangle \tag{9}$$

is finite although the expression is proportional to Z when expressed in terms of renormalized quantities. However, because of charge conservation all intermediate states in (9) contribute equally and not only the one particle state, so that the factor Z is exactly compensated. Correspondingly there is no infinite quenching of the interaction and no charge renormalization. Hence

([8]) T. D. Lee: *Phys. Rev.*, **95**, 1329 (1954).

([9]) See, F. Low: *Phys. Rev.*, **97**, 1392 (1955); H. Lehmann, K. Simanzik and W. Zimmermann: *Nuovo Cimento*, **1**, 205 (1955).

there is no necessity in this theory to introduce negative probabilities ([10]) and ghost states to get a finite interaction. This situation resembles the neutral scalar non-recoil theory where Z_2 also goes to zero but the theory ([8]) gives still a finite interaction potential between particles since there is no charge renormalization, It is therefore to be expected that relativistic theories with an infinite charge renormalization will behave in this respect more like the Lee-model than this theory. Nevertheless, the model shows that at least in one dimension there is a solution to the general equations of Lehmann, Symanzik and Zimmermann, which is not just the free fields. This excludes all possibilities of proving generally that they are consistent with the free fields only.

* * *

The author is indebted to Dr. GLASER for communicating to him his results before publication.

([10]) W. PAULI and G. KÄLLÉN: *Dans. Vid. Selsk.*, **30**, 23 (1955).

RIASSUNTO (*)

Si discute il significato fisico della soluzione esatta di campi spinoriali interagenti in uno spazio unidimensionale. Si ritiene che l'assenza di fantasmi in tale teoria sia connessa al fatto che si ha solo un campo ma non rinormalizzazione infinita della carica.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

Über den Anisotropen Charakter der Verallgemeinerten Relativistischen Räume.

A. Weissmann

Lehrstuhl für theoretische Physik der Bólyai - Universität Cluj Klausenburg, Rumänische Volksrepublik

(ricevuto il 25 Maggio 1958)

Zusammenfassung. — Unter dem Begriff « Raum » verstehen wir — in Übereinstimmung mit der Relativitätstheorie — immer den reellen, physikalischen Raum. Dieser ist nichts anderes, als der adäquate Ausdruck sämtlicher Eigenschaften des materiellen « Inhalts » der physikalischen Welt, d.h. die materielle Form selbst. Die Untersuchung der physikalischen Eigenschaften des so aufgefaßten Raumes erfolgt unter Zuhilfenahme der bekannten geometrischen Varietäten.

Unter dem Begriff « Raum » verstehen wir — in Übereinstimmung mit der Relativitätstheorie — immer den reellen, physikalischen Raum [1]. Dieser ist nichts anderes, als der adäquate Ausdruck sämtlicher Eigenschaften des materiellen « Inhalts » der physikalischen Welt, d.h. die materielle Form selbst.

1. – Erste Definition. *Laut* Schouten [2] *und* Eisenhart [3] *nennen wir verallgemeinerter relativistischer Raum eine solche — durch die* x^μ $(\mu = 1, 2, \ldots, 4)$ *Koordinaten bestimmte — vierdimensionale Varietät, in der ein vollkommen willkürlicher Tensor definiert wird:*

$$(1) \qquad {}^*g^{\mu\nu} = [g^{\mu\nu} - \varepsilon q^{\mu\nu}]$$

wo ${}^*g^{\mu\nu} \neq {}^*g^{\nu\mu}$ und $|{}^*g| \neq 0$.

(1) A. Weissmann: *Bólyai Tudományegyetem Évkönyv*, p. 33 (1955).
(2) J. A. Schouten: *Der Ricci-Kalkül.* (Berlin, 1924).
(3) L. P. Eisenhart: *Proc. of Nat. Ac. of U.S.A.*, p. 396 (1953).

Weiterhin $g^{\varrho\nu}g_{\mu\nu}=\delta^{\varrho}_{\cdot}$, wo $g_{\mu\nu}$ der Riemannsche fundamentale Tensor ist, $\varepsilon\ll 1$ einen beliebigen, ständigen Parameter bildet, während $\varphi^{\mu\nu}$ vorläufig vollkommen beliebige, nichtsymmetrische tensoriale Mengen zweiter Ordnung darstellen. Diese interpretieren wir weiter unten.

Erstes Postulat: Zwischen dem Tensor ${}^*g^{\mu\nu}$ und dem Konnexionskomponenten des verallgemeinerten Raumes ${}^*\Gamma^{\lambda}_{\nu\mu} \neq {}^*\Gamma^{\lambda}_{\mu\nu}$ postulieren wir folgende Relation:

$$\frac{\partial {}^*g^{\mu\nu}}{\partial x^{\lambda}} + {}^*g^{\varkappa\nu}\,{}^*\Gamma^{\mu}_{\varkappa\lambda} + {}^*g^{\mu\varkappa}\,{}^*\Gamma^{\nu}_{\varkappa\lambda} = 0\,. \tag{2}$$

In diesem Falle existiert nur ein einziger Tensor ${}^*g_{\mu\nu}=[g_{\mu\nu}+\varepsilon\varphi_{\mu\nu}]$, so daß ${}^*g_{\mu\nu}\,{}^*g^{\mu\nu}\approx g_{\mu\nu}g^{\mu\nu}=4$ richtig sei, wodurch wir die Möglichkeit des Herunter- und Heraufziehens der Indizes in die oben definierten verallgemeinten relativistischen Räume eingeführt haben.

Wie bekannt ([4]), ist die Bestimmung der affinen Konnexion ${}^*\Gamma^{\lambda}_{\mu\nu}$ auch im Falle der ganz allgemeinen Konnexionen mit vollkommener Eindeutigkeit, allein vermittels des ${}^*g_{\mu\nu}$ Tensorfeldes, möglich.

Nach obigen Ausführungen sei es uns gestattet, diese Methoden « Variationen der Metrik » zu nennen, da sie nichts anderes darstellen, als eine Verallgemeinerung der durch EINSTEIN eingeführten, analogen Methoden ([5]). Die so gewonnenen verallgemeinerten relativistischen Räume sind als « deformierte » Riemannsche Räume aufzufassen ([6]).

Nach verhältnismäßig einfachen Rechnungen gelangen wir zu nachstehender Relation:

$${}^*\Gamma^{\lambda}_{\nu\mu} = \Gamma^{\lambda}_{\nu\mu} + \frac{\varepsilon}{2}\left\{g^{\lambda\varkappa}\,\partial_{|\varkappa}\varphi_{\nu\mu} - 2\varphi^{\lambda}_{\varkappa}\Gamma^{\alpha}_{\nu\mu}\right\} - S^{\lambda}_{\cdot\,\mu\nu} - S^{\lambda}_{\cdot\,\nu\mu} - S_{\mu\nu}^{\cdot\,\cdot\,\lambda}, \tag{3}$$

wo $\Gamma^{\lambda}_{\nu\mu}$ Christoffel-Symbole sind, welche wir mit $g_{\mu\nu}$ berechnen. Die Bedeutung des Operators $\partial_{|\varkappa}$ ist:

$$\partial_{|\varkappa}\varphi_{\nu\mu} = \left\{\frac{\partial\varphi_{\varkappa\mu}}{\partial x^{\nu}} + \frac{\partial\varphi_{\nu\alpha}}{\partial x^{\mu}} - \frac{\partial\varphi_{\nu\mu}}{\partial x^{\varkappa}}\right\}, \tag{3a}$$

während: $S_{\nu\mu}^{\cdot\,\cdot\,\lambda} = \frac{1}{2}({}^*\Gamma^{\lambda}_{\nu\mu} - {}^*\Gamma^{\lambda}_{\mu\nu})$.

Die tensoriellen Mengen $\varphi_{\mu\nu}$ nennen wir Tensor der Funktion des Energie-Impulses. Diesen Tensor interpretieren wir als die vierdimensionale, nicht-

([4]) M. M. TONNELAT: *Journ Phys. et Rad.*, **16**, 21 (1955).
([5]) A. EINSTEIN: *Berl. Ber.*, p. 688 (1916).
([6]) A. WEISSMANN: *Studii si cercetări, Cluj*, no. 1-2, p. 101 (1956).

symmetrische Verallgemeinerung der Airyschen Spannungsfunktionen (7), die zur Bestimmung des Massentensors $T_{\mu\nu}$ dienen, welcher die Zustandsgleichung der deformierbaren, kontinuierlichen Medien befriedigt.

Wie bekannt, ist nämlich in der Theorie der deformierbaren, kontinuierlichen Medien bewiesen, daß unter den Bedingungen der kontinuierlichen Deformation, der Rotor der transponierten Rotation des Deformationstensors ($\mathcal{D}^*$ ist als Summe dreier Dyaden aufzufassen) gleich Null ist:

$$Q_{\mu\nu} = \overbrace{\text{rot}\,(\text{rot}\,\mathcal{D}^*)} = 0\,. \tag{4}$$

Die Relation (4) — welche eigentlich die Saint-Venantschen Kompatibilitätsbedingungen ausdrückt — ist auch in nachstehender Form aufschreibbar:

$$R_{\mu\nu\sigma\varrho} = \frac{1}{2}\left(\frac{\partial^2 {}^*g_{\varrho\sigma}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 {}^*g_{\mu\nu}}{\partial x^\varrho \partial x^\sigma} - \frac{\partial^2 {}^*g_{\mu\sigma}}{\partial x^\nu \partial x^\varrho} - \frac{\partial^2 {}^*g_{\nu\varrho}}{\partial x^\mu \partial x^\sigma}\right) + \tag{5}$$

$$+ {}^*\Gamma^\alpha_{\mu\nu}\,{}^*\Gamma_{\varrho\sigma,\alpha} - {}^*\Gamma^\alpha_{\mu\sigma}\,{}^*\Gamma_{\varrho\nu,\alpha} \equiv 0\,.$$

Diese Relation bildet die notwendige und hinreichende Bedingung dafür, daß es eine Punkttransformation gibt, die den Übergang von der inneren Metrik ${}^*g_{\mu\nu}$ zur Euklidischen Metrik ermöglicht.

Weiterhin ist bekannt, daß der Massen-Energie-Impuls Tensor $T_{\mu\nu}$, welcher in den Einsteinschen Gleichungen:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = k T_{\mu\nu}\,, \tag{6}$$

rechts figuriert, und welcher gleichzeitig die Zustandsgleichung der deformierbaren, kontinuierlichen Medien:

$$\frac{1}{\sqrt{g}}\,\frac{\partial \sqrt{g}\, T_{\mu\nu}}{\partial x^\nu} + \frac{1}{2}\,\frac{\partial g^{\lambda\nu}}{\partial x^\mu}\,T_{\lambda\nu} = 0\,, \tag{7}$$

befriedigt, als Rotor der transponierten Rotation des von Airy (7) eingeführten Tensors der Spannungsfunktion aufgefaßt werden kann:

$$(T_{\mu\nu}) = \mathcal{C} = \text{rot}\,\overbrace{(\text{rot}\;\varphi)}\,. \tag{8}$$

Das Obengesagte bildet den fundamentalen und universalen Zusammenhang zwischen der inneren Metrik der deformierbaren kontinuierlichen Medien und den Spannungsfunktionen, und liegt der vierdimensionalen, nichtsymmetrischen Verallgemeinerung zu Grunde.

(7) G. B. Airy: *Phil. Trans.*, **53**, 49 (1863).

Mit den, aus der allgemeinen Relativitätstheorie bekannten Methoden, unter Verwendung der Werte (1) und (3) berechnen wir die Mengen ${}^*R_{\omega\ \mu\nu}^{\ \ \lambda}$, ${}^*R_{\mu\nu}$ und ${}^*R$.

Nun können wir die Feldgleichungen in den verallgemeinerten relativistischen Räumen aufstellen:

$$k\,{}^*T_{\mu\nu} = {}^*G_{\mu\nu} \tag{9}$$

wo

$$\begin{aligned} {}^*G_{\mu\nu} &= {}^*R_{\mu\nu} - \frac{1}{2}\,{}^*g_{\mu\nu}{}^*R = R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R + \\ &+ \frac{1}{2}\,Z_{\mu\nu} + \frac{\varepsilon}{2}\left\{\frac{1}{2}\,F_{\mu\nu} - \varphi_{\mu\nu}(R + g^{\mu\nu}Z_{\mu\nu}) + g_{\mu\nu}\varphi^{\mu\nu}(R_{\mu\nu} + Z_{\mu\nu})\right\}, \end{aligned} \tag{10}$$

$R_{\mu\nu}$ den Ricci-Tensor, und R den Riemannschen Invariant darstellt. Mit $Z_{\mu\nu}$ haben wir die $S_{\mu\nu}^{\ \ \lambda}$ Felder enthaltenden Glieder und mit $F_{\mu\nu}$ die in ${}^*\Gamma_{\mu}^{\lambda}$ befindlichen $\{...\}$ Klammern enthaltenden Glieder bezeichnet.

Aus obigem geht hervor, daß ${}^*G_{\mu\nu}$ aus zwei Teilen besteht: erstens aus $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, welche die aus den Einsteinschen Gravitationsgleichungen bekannten Riemannschen Glieder darstellen, und zweitens aus ${}'G_{\mu\nu} = \frac{1}{2}Z_{\mu\nu} + (\varepsilon/2)\{...\}$, welche einige der Eigenschaften der verallgemeinerten Räume ausdrücken, die von denen der Riemannschen Räume abweichen. Wie ersichtlich, wird dies Glied von den tensoriellen Mengen $\varphi_{\mu\nu}$ bestimmt.

Die in den Riemannschen Räumen gültigen Einsteinschen Gleichungen beschreiben nur die Gravitationseigenschaften der Materie (da diese Theorie eben im Wesentlichen die Theorie der Schwerkraft ist). Eben darum können wir den in (9) auftretenden Energie-Impuls-Tensor $T_{\mu\nu}$ als aus zwei Gliedern bestehend auffassen: ${}^*T_{\mu\nu} = T_{\mu\nu} + \tau_{\mu\nu}$. Hier haben wir mit $T_{\mu\nu}$ den Teil des Massentensors der Materie bezeichnet, der ausschließlich von den Spannungen mit Gravitationscharakter bestimmt wird. Die Spannungen anderer Natur bestimmen $\tau_{\mu\nu}$.

Nun zerfällt die Gleichung (9) in zwei Teile:

$$kT_{\mu\nu} = G_{\mu\nu}\,, \tag{11a}$$

$$k\tau_{\mu\nu} = {}'G_{\mu\nu} = \tfrac{1}{2}Z_{\mu\nu} + (\varepsilon/2)\{...\}\,, \tag{11b}$$

wo die Gleichung (11b) die nicht schwerkraftlichen physikalischen Eigenschaften der verallgemeinerten Räume ausdrückt.

Es ist leicht ersichtlich, daß diese verallgemeinerten Räume als « deformierte » Riemannsche Räume aufgefaßt werden müssen. Weiterhin ist ersichtlich, daß diese « Deformation » von den Spannungen des Raumes ($\tau_{\mu\nu}$), die nicht Gravitationscharakter haben, ausgelöst worden sind.

Diese « Deformation » des Riemannschen Raumes ist die direkte Verallgemeinerung der Cartanschen Deformation des Euklidischen Raumes. CARTAN (8) sagt: « ... aus dem Gesagten geht hervor, daß wir den Zustand eines Mediums — das sich im Gleichgewicht befindet — so darstellen können, daß wir annehmen, der Raum — welcher dies Medium „enthält" — habe sich deformiert, und daß eben dieser Spannungszustand des Mediums diese geometrische Deformation physikalisch zum Ausdruck bringt ».

2. – Unser Ziel ist es zu demonstrieren, daß diese « Deformation » des Raumes sowohl im allgemeinen, wie auch im Riemannschen Speziallfall den anisotropen Charakter des in Betracht gezogenen physikalischen Raumes widerspiegelt.

Zweite Definition: *Unter Anisotropie verstehen wir das in verschiedenen Richtungen nich äquivalente Verhalten eines Mediums (physikalisches Kontinuum – Raum) gegenüber beliebigen physikalischen Einwirkungen.*

Dritte Definition: *Unter Inhomogenität verstehen wir das an verschiedenen Orten nicht äquivalente Verhalten eines Mediums (physikalisches Kontinuum – Raum) gegenüber beliebigen physikalischen Einwirkungen.*

Diese beiden physikalischen Eigenschaften sind nicht gleichbedeutend, doch besteht zwischen ihnen eine enge Beziehung:

Vierte Definition: *Die auf einem bestimmten Grad geordnete Inhomogenität ergibt Anisotropie.*

Auf Grund der Definitionen 2, 3 und 4 ist die Anisotropie des Mediums immer vermittels lokaler Inhomogenitäten, d.h. vermittels Ortsfunktionen zu charakterisieren.

Die Definition der im Sinne Levi-Civitas aufgefaßten Parallelverschiebung ist bekannt (1):

$$\delta A_\mu \overset{\text{def}}{=} \frac{1}{2}\iint {}^*R_{\mu\nu\sigma\varrho} A^\varrho \, \mathrm{d}S^{\nu\sigma}, \tag{12}$$

wo ${}^*R_{\mu\nu\sigma\sigma}$ den Krümmungstensor darstellt, während

$$\mathrm{d}S^{\nu\sigma} \overset{\text{def}}{=} \mathrm{d}x^\nu \, \delta x^\sigma - \mathrm{d}x^\varrho \, \delta x^\nu = \begin{vmatrix} \mathrm{d}x^\nu & \mathrm{d}x^\sigma \\ \delta x^\nu & \delta x^\sigma \end{vmatrix}, \tag{12a}$$

ein antisymmetrischer Tensor ist, der das auf die $\mathrm{d}x^\nu$, δx^σ infinitesimalen Wege aufgespannte Flächenelement definiert. Die Gleichung bestimmt die Änderung

(8) E. CARTAN: *Compt. Rend.*, **174**, 437 (1922).

des verschobenen Vektors, d.h. die Änderung der behandelten physikalischen Wirkung. Die rechte Seite der Gleichung (12) ist ein Flächenintegral. Zweifellos ist in diesem der Vektor A^ϱ ortsabhängig. Sofern der fundamentale Tensor ${}^*g_{\mu\nu}$ ortsabhängig ist, ist auch der Krümmungstensor ${}^*R_{\mu\nu\sigma\varrho}$ ortsabhängig. *Die Änderung des Vektors δA_μ ist jedoch schon richtungsabhängig*, und zwar hängt sie von der räumlichen Richtung und dem Sinn der Normalen der durchlaufenen Fläche ab. In vier Dimensionen kann dies nicht so augenscheinlich dargestellt werden, doch drückt die Antisymmetrie des Tensors $dS^{\nu\sigma}$ eben dies aus.

Das Integral (12) welches im gegebenen Raum die Änderung irgendeiner physikalischen Wirkung in der Funktion der Bewegung charakterisiert, ist richtungsabhängig, d.h. in verschiedenen Richtungen ist sein Verhalten nicht äquivalent, also stellt es eben den anisotropen Charakter des in Betracht gezogenen Raumes dar. Der Tensor ${}^*R_{\mu\nu\sigma\varrho}$ widerspiegelt den für jeden einzelnen Fall konkretisierten Charakter der Anisotropie.

Im Euklidischen Raum ${}^*R_{\mu\nu\sigma\varrho} = 0$. Der Vektor A_μ ändert sich nicht. Das Medium (physikalisches Kontinuum – Raum) ist isotrop.

Im Riemannschen Raum entspricht die Änderung einer Rotation, welche nicht nur von der Form der durchlaufenen Kontur, sondern auch von der Richtung der Fläche — welche auf diese Kontur aufgespannt ist — abhängt. Diese Rotation hängt also auch von den Raumrichtungen ab. Die physikalische Wirkung ändert sich somit je nach der Richtung. Dies also bedeutet für die Komponenten des verschobenen Vektors ein nicht äquivalentes Verhalten. Um von einer Analogie aus der Physik der elastischen Körper Gebrauch zu machen: die oben gezogenen Folgerungen entsprechen der Anisotropie der Kristalle, die dem Hookeschen Gesetz mit konstanten Modulen unterworfen sind.

Im Falle Weylscher Räume, entspricht der Änderung des Vektors A_μ ausser einer Rotation, auch noch eine Homothetie (Dehnung im Sinne zentroaffiner Transformation). Die physikalische Wirkung ändert sich je nach der Richtung und zeigt zugleich, bezüglich ihrer Größe, noch eine Ortsabhängigkeit. Diese bedeuten das nicht äquivalente Verhalten der Richtungen auch bezüglich der Eichung. Dies ist Anisotropie im Sinne solcher — dem verallgemeinten Hookeschen Gesetz unterworfenen — Medien, in denen die Elastizitätsmodulen Ortsfunktionen sind. Die Richtungsabhängigkeit und die Ortsabhängigkeit stehn hier jedoch noch in keinem expliziten funktionalen Zusammenhang.

Nachstehende Verallgemeinerung ist auch in den nichtsymmetrischen Räumen gültig, wie z.B..

In Torsionsräumen, wo die Änderung des Vektors A_μ, neben der Rotation und affinen Verschiebung, auch noch einer Translation entspricht. Die anisotropie entspricht hier solchen Medien, in denen, ausser den obigen, auch noch eine Torsionswirkung zur Geltung kommt.

Wie aus den auf Grund der unter 1. gegebenen Definition und Postulates

durchgeführten Rechnungen hervorgeht, besteht der Tensor ${}^*R_{\mu\nu\sigma\varrho}$ tatsächlich aus zwei Teilen: dem Riemann-Christoffelschen Tensor $R_{\mu\nu\sigma\varrho}$ — den wir auf Grund der Christoffelschen Konnexionskomponenten $\Gamma_{\mu\nu}$ berechnet haben, und dem Tensor ${}'R_{\mu\nu\sigma\varrho}$, welchen die Mengen $\varphi_{\mu\nu}$ bestimmen.

Ziehen wir all dies in Betracht, so folgt aus der Gleichung (12) sofort der anisotrope Charakter der verallgemeinten relativistischen Räume.

3. – Um diesem Zusammenhang zwischen der Anisotropie des Raumes in unserem Sinne und der Anisotropie im Sinne der Mechanik der elastischen Körper unmittelbar vorzuführen, kehren wir zu einem Spezialfall des unter **1.** angewandten Algorithmus zurück.

Führen wir die sogenannte schwache Variation ein:

$$(13) \qquad {}^*g_{\mu\nu} = [\overset{\circ}{g}_{\mu\nu} + \varepsilon\varphi_{\mu\nu}]$$

wo: $\overset{\circ}{g} = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases}$ und $\varphi_{\mu\nu} = \varphi_{\nu\mu}$.

Nach den, den obigen vollkommen entsprechenden Rechnungen erhalten wir den Tensor ${}^*G_{\mu\nu}$, welcher aus zwei Teilen besteht:

$$(14) \qquad {}^*G_{\mu\nu} = \overset{\circ}{G}_{\mu\nu} + G_{\mu\nu},$$

wo $G_{\mu\nu}$ offensichtlich mit 0 identisch ist und die Euklidischen Eigenschaften des Raumes widerspiegelt. $G_{\mu\nu}$ drückt die von den Euklidischen Eigenschaften abweichenden Eigenschaften des « deformierten » Raumes aus.

Durch Einführung von (14) in die Feldgleichungen (9) zerfallen diese in je zwei Teile. Bei der in zwei Teile erfolgenden Spaltung des linksstehenden Massentensors müssen wir gewisse physikalische Gesichtspunkte in Betracht ziehen, und zwar, daß der Euklidische Raum — wie dies aus der Relativitätstheorie bekannt ist — einem « leeren », von inneren Spannungen freien, kontinuierlichen Medium entspricht.

Es ist bekannt, daß die Deformation der inneren Metrik dieses leeren Kontinuums die Saint-Venantschen Kompatibilitätsbedingungen (4)–(5) befriedigen muß, d.h. die deformierte Metrik bleibt auch weiterhin eine Euklidische Metrik.

Aus der Theorie der deformierbaren Medien wissen wir jedoch, daß dieser Fall dem einfachen Hookeschen Gesetz entspricht. Dieses kann in folgender Form aufgeschrieben werden:

$$(15) \qquad \tau_{\mu\nu} = (\lambda\overset{\circ}{g}_{\mu\nu}\overset{\circ}{g}_{\alpha\beta} + 2\mu\overset{\circ}{g}_{\mu\alpha}\overset{\circ}{g}_{\nu\beta})\mathcal{D}^{\alpha\beta},$$

wo $\mathcal{D}^{\alpha\beta}$ den Deformationstensor der inneren Metrik des Raumes darstellt. Da $g_{\mu\nu}$ ein kugelsymmetrischer Tensor ist, folgt, daß die Hauptachsen der

Tensorflächen von $\tau_{\mu\nu}$ und $\mathcal{D}^{\nu\beta}$ zusammenfallen, was eben den isotropen Charakter des Euklidischen Raumes demonstriert.

Ziehen wir jedoch in Betracht, daß jede tatsächliche Variation der Euklidischen Metrik sofort das Auftreten der Mengen $\varphi_{\mu\nu}$ im Krümmungstensor bedingt, folgt, daß diese reelle « Deformation » des Raumes von seinem materiellen « Inhalt » hervorgerufen wird und so ist der deformierte Raum kein Euklidischer Raum mehr.

Hierbei nämlich tritt an Stelle von (4) bzw. (5)

$$Q_{\mu\nu} \neq 0 \; ; \qquad R_{\mu\nu\sigma\varrho} \neq 0 \tag{16}$$

auf. Trotzdem besteht auch weitherhin der in (4) und (8) aufgestellte Zusammenhang zwischen dem Deformationstensor (und der inneren Metrik) einerseits und dem Tensor $\varphi_{\mu\nu}$ adrerseits.

In diesem Falle aber ist — bei einer Deformierung — in der infinitesimalen Umgebung eines Punktes nachstehende allgemeine Form der Affintransformation gültig:

$$\mathrm{d}\overset{*}{x}^{\mu} = \varphi_{\nu}^{\cdot\mu}\, \mathrm{d}x^{\nu} \, . \tag{17}$$

In vollkommener Analogie hierzu, ist die allgemeine Transformation der Komponenten des metrischen Tensors vom deformierbaren Medium (Raum):

$$^{*}g_{\mu\nu} = \alpha_{\mu}^{\;i}\, \alpha_{\nu}^{\;k}\, g_{ik} \tag{18}$$

(wo $\alpha_{\mu}^{\;i}$ und $\alpha_{\nu}^{\;k}$ beliebige Punktfunktionen darstellen).

Selbstverständlich ist die (17) Transformation im allegmeinen nicht holonom, während die (18) Transformation eine nicht Euklidische Metrik zustandebringt.

Unter diesen Bedingungen besteht zwischen dem Spannungstensor $\tau_{\mu\nu}$ und dem Deformationstensor $\mathcal{D}^{\nu\beta}$ selbstverständlich auch weiterhin eine invariante Beziehung, die als Verallgemeinerung des Hookeschen Gesetzes (15) aufgefaßt werden kann. In dieser Relation aber treten an Stelle der konstanten Skalaren Koeffizienten — eben infolge des Vorhandenseins von $\varphi_{\mu\nu}$ — Tensoren höherer Ordnung auf, welche das in verschiedenen Richtungen nicht äquivalente Verhalten unseres Mediums (des Raumes) kennzeichnen. Der einfachste mögliche Zusammenhang zwischen $\tau_{\mu\nu}$ und $\mathcal{D}^{\nu\beta}$ ist:

$$\tau_{\mu\nu} = {}^{*}C_{\mu\nu\varkappa\beta}\, \mathcal{D}^{\varkappa\beta} \, . \tag{19}$$

Aus dem Charakter von $^{*}C_{\mu\nu\varkappa\beta}$ folgt, daß die Relation (19) — mit Ausnahme des in (15) gezeigten Falles — im allgemeinen nicht mehr zur Koinzidenz der Hauptachsen von $\tau_{\mu\nu}$ und $\mathcal{D}^{\nu\beta}$ führt, was den anisotropen Charakter dieser « deformierten » Räume unmittelbar zeigt.

Die Dissertationsarbeit enthält noch weitere detaillierte Rechnungen bezüglich mehrerer Spezialfälle aus der allgemeinen Abhandlung **1.**, wie z.B. hinsichtlich des anisotropen Charakters der Riemannschen, Weylschen und Eddingtonschen Räume.

4. – Unser Ziel ist, die Bestimmung des Anisotropietensors ${}^*C_{\mu\nu\alpha\beta}$ und das Ausweisen seiner Verbindung mit dem fundamentalen Tensor der verallgemeinten Räume.

Zu diesem Zweck bilden wir den Skalaren:

$$\gamma = {}^*C_{\mu\nu\alpha\beta} x^\mu x^\nu x^\alpha x^\beta \tag{20}$$

unter Benützung der Matrix des Elastizitätsmoduls der kontinuierlichen Medien. (20) bildet eine Hyperfläche vierter Ordnung mit Zentrum.

Betrachten wir diese Hyperfläche als eine in einem Euklidischen Raum, entsprechender Dimension, immersionierte allgemeine Affinvarietät. Die Dimension unseres Euklidischen Raumes wird von den sogenannten verallgemeinerten Gauß-Codazzischen und Ricci-Kühneschen [1] Bedingungen gegeben. Im Falle eines in E_n immersionierten A_m — wenn wir die Komponenten des Krümmungstensors der induzierten Konnexion ${}^*\Gamma^\sigma_{\mu\nu}$ mit ${}^*R^\sigma_{\mu\nu\varrho}$ bezeichnen — ist die erste dieser Beziehungen

$${}^*R^\sigma_{\mu\nu\varrho} + {}^*D^\sigma_{i\nu}\,{}^*D^i_{\mu\varrho} - {}^*D^\sigma_{i\varrho}\,{}^*D^i_{\mu\nu} = 0\,. \tag{21}$$

Hier bilden die Mengen ${}^*D^i_{\mu\nu}$ die den A_m Räumen zugeordneten Koeffizienten der sogenannten « fundamentalen Form zweiter Ordnung ». Diese Zuordnung erfolgt in vollkommener Analogie zu dem bei den Riemannschen Räumen angewandten Gedankengang.

Unter Verwendung der Erwägungen von A. Friedman [9], gelangen wir nun zu den verallgemeinten vierdimensionalen relativistischen Räumen, die Gegenstand unserer Untersuchung bilden.

Nehmen wir als Metrik eines fünfdimensionalen Euklidischen Raumes:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu - c^2\,\mathrm{d}t^2 \qquad (\mu, \nu = 1 \ldots 4) \tag{22}$$

an. In diesem Raum drückt die Gleichung:

$$S_{\mu\mu} X^\mu X^\mu - K = 0 \tag{23}$$

[9] A. Friedmann: *Zeits. f. Phys.*, **10**, 377 (1922).

einen zur Achse der Zeit parallelen, «allgemeinen Zylinder» aus. Die räumliche Sektion dieses Zylinders wird eben der der Gleichung (20) entsprechende vierdimensionale, Riemannsche Raum sein. Auf Grund eines solchen Immersionsprinzips erhalten wir auch unseren vierdimensionalen verallgemeinerten relativistischen Raum. Nachdem wir die Gleichung der Hyperfläche (20), unter Anwendung mehrer Transformationen, parametrisieren und die Koeffizienten der fundamentalen Formen erster und zweiter Ordnung auf bekannte Art berechnen ([1]), gehen wir zur Herstellung von ${}^*C_{\mu\nu\alpha\beta}$ über. Hierzu bedienen wir uns der Überlegungen von J. B. RUMER ([10]). Ihm ist es gelungen, aus den Koeffizienten der (Riemannschen) fundamentalen Form erster Ordnung ($g_{\mu\nu}$) und aus denen der (Riemannschen) fundamentalen Form zweiter Ordnung ($D^i_{\mu\nu}$) den $\tau^i_{\mu\nu}$ Tensor zu konstruieren, anhand der Relation:

$$\tau^i_{\mu\nu} = D^i_{\mu\nu} - g_{\mu\nu} D^i \,. \tag{24}$$

Dank dieser Relation, hat er aus den Gauß-Codazzischen Gleichungen ein, den Einsteinschen Gleichungen gleichwertiges, System gebildet. Weitherhin beweist er, daß die Raumtheorie in dieser Form der Elastizitätstheorie der kontinuierlichen Medien vollkommen analog ist. Die Quantitäten $D^i_{\mu\nu}$ verhalten sich wie vierdimensionale Deformationen, die Quantitäten $\tau^i_{\mu\nu}$ wie vierdimensionale Spannungen und (24) ersetzt das Hookesche Gesetz.

Führen wir nun in die Rumersche Gleichungen die von uns angewandten Mengen:

$$\begin{cases} {}^*g^{\mu\nu} = [g^{\mu\nu} - \varepsilon\varphi^{\mu\nu}] \,, \\ {}^*g_{\mu\nu} = [g_{\mu\nu} + \varepsilon\varphi_{\mu\nu}] \,, \end{cases} \tag{25}$$

ein, erhalten wir aus (21) nach Multiplikation und Kontraktion:

$${}^*R_{\mu\nu} - \tfrac{1}{2}\,{}^*g_{\mu\nu}\,{}^*R = ({}^*D^i_{\mu\nu}\,{}^*D_i - {}^*D^i_{\mu\varrho}\,{}^*D^\varrho_{i\nu}) - \tfrac{1}{2}\,{}^*g_{\mu\nu}({}^*D^i\,{}^*D_i - {}^*D^i_{\nu\varrho}\,{}^*D_i{}^\varrho) \,. \tag{26}$$

Stellen wir die Gleichung

$$\tau^i_{\mu\nu} = {}^*D^i_{\mu\nu} - {}^*g_{\mu\nu}\,{}^*D^i \tag{27}$$

auf, so kann deren rechte Seite — unter Zuhilfenahme von (25) — infolge Indexspiels und Kontraktion, in nachstehende Form gebracht werden:

$$\tau^i_{\mu\nu} = \left[\tfrac{1}{2}({}^*g_{\mu\varkappa}\,{}^*g_{\nu\beta} + {}^*g_{\mu\beta}\,{}^*g_{\nu\varkappa}) - D\,{}^*g_{\mu\nu}\,{}^*g_{\varkappa\beta}\right]\mathcal{D}^{\varkappa\beta,i} \,. \tag{28}$$

([10]) J. B. RUMER: *Göttinger Nachrichten*, S. 148 (1931).

(Hier bildet D einen dimensionalen Faktor.) Aus dem Vergleich von (19) folgt:

$$(29) \qquad {}^*C_{\mu\nu\varkappa\beta} = \tfrac{1}{2}({}^*g_{\mu\varkappa}{}^*g_{\nu\beta} + {}^*g_{\mu\beta}{}^*g_{\nu\varkappa}) - D^*g_{\mu\nu}{}^*g_{\varkappa\beta}\,.$$

Die weitere Untersuchung des Ergebnisses (29) führt zur Annahme ([11]), daß die elastischen Eigenschaften der technischen Materien — die von polykrystalliner Struktur sind — vielleich mit dem mathematischen Apparat der verallgemeinerten relativistischen Räume behandelt werden könnten.

([11]) A. WEISSMANN: *Studii si cercetări. Iasi, seria fiz-tehn.*, no. 1-2, p. 3 (1956).

RIASSUNTO (*)

Come «spazio» comprendiamo — d'accordo con la teoria della relatività — sempre lo spazio fisico reale. Questo non è che l'espressione adeguata di tutte le proprietà del «contenuto» materiale del mondo fisico, cioè la stessa forma materiale. L'esame delle proprietà fisiche dello spazio così concepito si esegue con l'ausilio delle note varietà geometriche.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

The Polarization of Cosmic Ray Muons (*+).

J. M. Fowler, H. Primakoff and R. D. Sard

Washington University - St. Louis. Mo., U.S.A.

(ricevuto il 29 Maggio 1958)

Summary. — Cosmic-rays muons of definite energy and direction of motion are expected to show a partial longitudinal polarization because of the falling energy spectrum of the parent pions. The general theory of muon polarization in $\pi \to \mu$ decay in flight is worked out, and applied to cyclotron as well as cosmic-ray experiments. Cosmic-ray muons are predicted to have a partial longitudinal polarization of about 23%. The theory is also applied to the $K \to \mu$ decay. An experiment has been carried out to test the prediction for cosmic-ray muons. Decay electrons from muons stopped in an aluminum absorber are detected in identical scintillation counters placed symmetrically above and below the absorber. The delay and amplitude of each observed pulse have been measured, and used to sort out the $\mu^+ \to e^+$ decays from the $\mu^- \to e^-$ and background contributions. The ratio of $\mu^+ \to e^+$ events detected in the upper and lower scintillators is 1.14 ± 0.07, as compared with a theoretical prediction of about 1.11. To test the symmetry of the apparatus the experiment has also been carried out with an absorber of salt, which is known to depolarize stopped muons almost completely. The upper to lower ratio is in this case 1.03 ± 0.06, to be compared with the theoretically expected value of about 1.02. It is concluded that the various assumptions made in the theory of muon polarization in $\pi \to \mu$ decay in flight are correct and that muons suffer negligible depolarization while losing 2 GeV energy in the atmosphere.

(*) Research supported by the joint program of the Office of Naval Research and the Atomic Energy Commission, and by the Air Force Office of Scientific Research of the Air Research and Development Command.

(+) These results were briefly summarized at the Padua-Venice Conference on Mesons and other Recently Discovered Particles, September, 1957 (mimeographed edition of *Proceedings*, p. IV-107).

1. – Introduction.

The spin polarization of muons produced in $\pi \to \mu$ decay is now well established. This polarization is close to 100 % for muons originating from $\pi \to \mu$ decay in flight of cyclotron produced pions (1). Since cosmic ray muons are predominantly due to $\pi \to \mu$ decay in flight in the upper atmosphere the question arises whether any polarization of the cosmic ray muon « beam » at sea level is to be expected (2).

A muon of given energy and direction of motion may be the decay product of parent pions of various energies and directions of motion — considering extreme cases, a relativistic muon of energy E is either emitted forward by a relativistic pion of energy $E' = 1.00\, E$ or backward by a relativistic pion of energy $E'' = 1.75\, E > E'$. Since the pion energy spectrum falls off sharply with increasing pion energy, and since the muon emission probability in the pion rest frame is isotropic (3), it is more probable for any given muon to have been emitted forward by a more frequent lower energy pion than backward by a less frequent pion of higher energy. Now to a sufficient approximation any emitted positive muon may be considered as completely polarized parallel (4), say, to its direction of motion in the parent pion rest frame. As a result the muon appears in the laboratory frame as also completely polarized, respectively parallel or antiparallel to its direction of motion for forward or backward emission by the parent pion (see Eq. (19) below). Any given muon is thus more likely upon production to possess a parallel rather than an antiparallel longitudinal polarization, and we anticipate that the cosmic ray muon « beam » is indeed partially longitudinally polarized near the top of the atmosphere. Neglecting for the time being any depolarization arising from elastic and inelastic collisions of the muons with air molecules, we may in turn anti-

(1) L. LEDERMAN, V. L. TELEGDI, M. KAPLON and others: *Proceedings of the 1957 Rochester Conference*, Chap. VII (New York, 1957); D. H. WILKINSON: *Nuovo Cimento*, **6**, 516 (1957).

(2) This question has been concurrently studied at M.I.T. with essentially the same results as ours and, in addition, with estimates of the muon depolarization in slowing down (S. HAYAKAWA: *Phys. Rev.*, **108**, 1533 (1957); G. W. CLARK and J. HERSIL: *Phys. Rev.*, **108**, 1538 (1957)). We are grateful to Professor CLARK for communicating the M.I.T. results to us before publication.

(3) R. LEVI-SETTI, R. GARWIN, A. ALIHANIAN, and others: *Proceedings of the Padua-Venice Conference* (1957) (mimeographed edition).

(4) It appears that the polarization of the positive muons is actually antiparallel (G. CULLIGAN, S. C. F. FRANK, J. R. HOLT, J. C. KLUYVER and T. MASSAM: *Nature*, **180**, 751 (1957); T. D. LEE: *Midwest Conference on Theoretical Physics, St. Louis* (1958) (unpublished)). This does not affect our arguments, which are concerned with the degree of polarization, not its sense.

cipate a partial longitudinal polarization for the cosmic ray muon « beam » with particular energy and direction of motion at sea level. Such a polarization can be observed by the emission asymmetry in the muon decay electrons, (see Eq. (13*b*) below) the muons themselves being stopped in a suitable non depolarizing moderator, such as aluminum. The experiment reported here describes preliminary evidence for the existence of this polarization. Since the magnitude of the polarization observed depends on the shape of the parent pion energy spectrum (see Eq. (24) below), conclusions may eventually be obtained from the study of the polarization regarding the shape of this spectrum in various ranges of pion energy.

2. – Theory.

Let the muon spin density (pseudo) four vector: $\{\text{Trace}\,(\varrho\boldsymbol{\sigma}), \text{Trace}\,(\varrho\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}/3)\}$ be denoted by $s_j = \{\boldsymbol{s}, s_4\}$ [5]. Here ϱ is the density-matrix operator describing the probability with which any particular muon occupies the various possible muon spin-direction states; $\boldsymbol{\sigma}$ and $\boldsymbol{\alpha}$ are the muon spin and muon velocity Dirac-matrix operators. In the muon rest frame we have:

$$s^{(\mu)} = \text{Trace}\,(\varrho^{(\mu)}\boldsymbol{\sigma}) = \boldsymbol{\xi}\,. \tag{1a}$$

(Superscripts are used throughout to label the reference frame). Here $\boldsymbol{\xi}$ is called the muon polarization vector; with suitable normalization $|\boldsymbol{\xi}| \leqslant 1$. The case $|\boldsymbol{\xi}| = 1$ corresponds to a muon completely polarized in its rest frame.

Eq. (1*a*) implies that $\varrho^{(\mu)} = \frac{1}{4}(1+\beta)(1+\boldsymbol{\xi}\cdot\boldsymbol{\sigma})$ where β is the Dirac-matrix operator associated with the muon rest energy. This value for $\varrho^{(\mu)}$ yields:

$$s_4^{(\mu)} = \text{Trace}\,(\varrho^{(\mu)}\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}/3) = 0\,. \tag{1b}$$

From Eqs. (1*a*), (1*b*) we may now prove that s_j is orthogonal to the muon momentum-energy four-vector

$$p_j = \{m\boldsymbol{p}, m\varepsilon\} = \left\{\frac{m\boldsymbol{v}}{\sqrt{1-v^2}}, \frac{m}{\sqrt{1-v^2}}\right\}, \tag{2a}$$

since,

$$\sum_{j=1}^{4} s_j p_j = \boldsymbol{s}\cdot\frac{m\boldsymbol{v}}{\sqrt{1-v^2}} - \frac{s_4 m}{\sqrt{1-v^2}} = \sum_{j=1}^{4} s_j^{(\mu)} p_j^{(\mu)} = 0\,. \tag{2b}$$

---

(5) A good discussion of spin density (pseudo) four-vectors and related matters is given by H. A. Tolhoek: *Rev. Mod. Phys.*, **28**, 277 (1956).

Eq. (2*b*) immediately gives:

$$s_4 = \boldsymbol{s}\cdot\boldsymbol{v}\,, \tag{3}$$

as a relation holding in an « unprimed » frame of reference in which the muon possesses a velocity $\boldsymbol{v} = \boldsymbol{v}_I v$ and an energy (in units of its rest mass) of ε. We also have:

$$\sum_{j=1}^{4} s_j s_j = |\boldsymbol{s}|^2 - (s_4)^2 = |\boldsymbol{s}|^2 - (\boldsymbol{s}\cdot\boldsymbol{v})^2 = \sum_{j=1}^{4} s_j^{(\mu)} s_j^{(\mu)} = |\boldsymbol{\xi}|^2\,, \tag{4}$$

Eq. (4) yielding, as the basic relation between s and ξ:

$$\boldsymbol{s} = \boldsymbol{\xi} + \left(\frac{1}{\sqrt{1-v^2}} - 1\right)\boldsymbol{\xi}\cdot\boldsymbol{v}_I\boldsymbol{v}_I = \boldsymbol{\xi} + (\varepsilon - 1)\boldsymbol{\xi}\cdot\boldsymbol{v}_I\boldsymbol{v}_I\,. \tag{5}$$

Similarly, in a « primed » frame in which the muon possesses a velocity $\boldsymbol{v}' = \boldsymbol{v}_I' v'$ and an energy (in units of its rest mass) of ε', we obtain

$$\boldsymbol{s}' = \boldsymbol{\xi} + (\varepsilon' - 1)\boldsymbol{\xi}\cdot\boldsymbol{v}_I'\boldsymbol{v}_I'\,. \tag{6}$$

We now consider a Lorentz transformation between the « unprimed » frame and the « primed » frame. Then, with $\boldsymbol{u} = \boldsymbol{u}_I u$ the velocity of the « unprimed » frame with respect to the « primed » frame, with $\eta = 1/\sqrt{1-u^2}$, and with $s_4 = \boldsymbol{s}\cdot\boldsymbol{v}$ (Eq. (3)),

$$\boldsymbol{s}' = \boldsymbol{s} + (\eta - 1)\boldsymbol{s}\cdot\boldsymbol{u}_I\boldsymbol{u}_I + \eta(\boldsymbol{s}\cdot\boldsymbol{v})\boldsymbol{u}\,, \tag{7}$$

and, analogously,

$$\boldsymbol{p}' = \boldsymbol{p} + (\eta - 1)\boldsymbol{p}\cdot\boldsymbol{u}_I\boldsymbol{u}_I + \eta\varepsilon\boldsymbol{u}\,, \tag{8}$$

$$\varepsilon' = \eta(\varepsilon + \boldsymbol{p}\cdot\boldsymbol{u})\,, \tag{9}$$

so that, dividing Eq. (8) by Eq. (9):

$$\boldsymbol{v}' = \frac{\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{u}_I\boldsymbol{u}_I + \eta^{-1}(\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{u}_I\boldsymbol{u}_I)}{1 + \boldsymbol{v}\cdot\boldsymbol{u}}\,. \tag{10}$$

From Eqs. (7), (10) we obtain the transformation law for the longitudinal component of the muon spin density:

$$\boldsymbol{s}'\cdot\boldsymbol{v}' = \{\text{right hand side of Eq. (7)}\}\times\{\text{right hand side of Eq. (10)}\} = \tag{11}$$
$$= \eta(\boldsymbol{s}\cdot\boldsymbol{v} + \boldsymbol{s}\cdot\boldsymbol{u})\,.$$

Eqs. (5), (6), (11) yield:

$$(12) \qquad (\boldsymbol{\xi}\cdot\boldsymbol{v}_I') = \frac{\eta v\varepsilon}{v'\varepsilon'}\left[(\boldsymbol{\xi}\cdot\boldsymbol{v}_I) + \left(\frac{1}{v\varepsilon}\right)(\boldsymbol{\xi} + (\varepsilon-1)\boldsymbol{\xi}\cdot\boldsymbol{v}_I\boldsymbol{v}_I)\cdot\boldsymbol{u}\right].$$

We next identify the « unprimed » frame with the parent pion rest frame, the « primed » frame with the laboratory frame; $\boldsymbol{u}$, η are then the pion velocity and the pion energy (in units of its rest mass) in the laboratory frame. Further, we note from Eqs. (5), (6) that

$$(13a) \qquad (\boldsymbol{\xi}\cdot\boldsymbol{v}_I') = \lim_{v\to 0}(\boldsymbol{s}'\cdot\boldsymbol{v}_I'); \qquad (\boldsymbol{\xi}\cdot\boldsymbol{v}_I) = \lim_{v'\to 0}(\boldsymbol{s}\cdot\boldsymbol{v}_I)\,,$$

so that the quantities $(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')$, $(\boldsymbol{\xi}\cdot\boldsymbol{v}_I)$ play the role of muon longitudinal polarizations in the laboratory and pion rest frames, respectively. The muon longitudinal polarization $(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')$ determines the angular distribution of decay electrons observed in the laboratory when the muons are stopped in a moderator and undergo μ-e decay; neglecting depolarization effects in the moderator, the decay electron angular distribution is of the familiar form:

$$(13b) \qquad 1 - (\boldsymbol{\xi}\cdot\boldsymbol{v}_I')(a)\cos\psi\,,$$

where the coefficient in the angular distribution, a, depends on the decay electron energy and ψ is the space angle between the muon velocity-direction on first entering the moderator, $\boldsymbol{v}_I'$, and the decay electron momentum-direction (see Eqs. (30)-(34) *et seq.*, below).

We now note that in the pion rest frame the muon spin density $\boldsymbol{s}$ and the muon velocity $\boldsymbol{v}$ must be parallel. This conclusion follows since in the pion rest frame there are only two vectors which can specify directions: $\boldsymbol{v}$ and the pion spin vector, and there is abundant evidence that the pion spin is zero. Then, from Eq. (5)

$$(14) \qquad \boldsymbol{s} = \varepsilon|\boldsymbol{\xi}|\boldsymbol{v}_I = \varepsilon\boldsymbol{\xi}\,,$$

so that (a): $\boldsymbol{s}$ and $\boldsymbol{\xi}$ are also parallel, and (b): the muon longitudinal polarization in the pion rest frame is given by:

$$(15) \qquad (\boldsymbol{\xi}\cdot\boldsymbol{v}_I) = |\boldsymbol{\xi}|.$$

It is clear from Eq. (15) that the condition for complete muon longitudinal polarization in the muon rest frame, $|\boldsymbol{\xi}| = 1$, coincides with that for complete muon longitudinal polarization in the pion rest frame, $(\boldsymbol{\xi}\cdot\boldsymbol{v}_I) = 1$. Eqs. (15), (14) and (12) yield the fundamental expression for the muon longitudinal

polarization in the laboratory frame in terms of the muon longitudinal polarization in the pion rest frame:

$$(\xi\cdot\boldsymbol{v}_I') = |\xi|\frac{\eta v\varepsilon}{v'\varepsilon'}\left(1+\frac{\boldsymbol{v}_I\cdot\boldsymbol{u}}{v}\right). \tag{16}$$

Alternative forms of $(\xi\cdot\boldsymbol{v}_I')$, obtained by combining Eq. (16) and Eqs. (10), (9), are:

$$(\xi\cdot\boldsymbol{v}_I') = |\xi|\frac{\boldsymbol{v}+\boldsymbol{v}_I\cdot\boldsymbol{u}}{|\boldsymbol{u}+\boldsymbol{v}\cdot\boldsymbol{u}_I\boldsymbol{u}_I+\eta^{-1}(\boldsymbol{v}-\boldsymbol{v}\cdot\boldsymbol{u}_I\boldsymbol{u}_I)|}, \tag{17}$$

and

$$(\xi\cdot\boldsymbol{v}_I') = |\xi|\left(\frac{1}{v'v}\right)(1-\eta/\varepsilon'\varepsilon). \tag{18}$$ [6]

Eq. (17) gives in a simple way the limiting value of $(\xi\cdot\boldsymbol{v}_I')$ for forward and backward emission of the muon in the pion rest frame:

$$\begin{cases} (\xi\cdot\boldsymbol{v}_I') = |\xi|, & \text{for } \boldsymbol{v}_I\cdot\boldsymbol{u} = u, \\ (\xi\cdot\boldsymbol{v}_I') = |\xi|, & \text{for } \boldsymbol{v}_I\cdot\boldsymbol{u} = -u, \text{ and } v>u, \\ (\xi\cdot\boldsymbol{v}_I') = -|\xi|, & \text{for } \boldsymbol{v}_I\cdot\boldsymbol{u} = -u, \text{ and } v<u. \end{cases} \tag{19}$$

The pion energies $\eta_{\min}$, $\eta_{\max}$ corresponding to $\boldsymbol{v}_I\cdot\boldsymbol{u} = \pm u$, and so corresponding to $(\xi\cdot\boldsymbol{v}_I) = \pm|\xi|$ are found from Eqs. (9), (10), or directly from Eq. (18), as

$$\begin{cases} \eta_{\max} = \varepsilon'\varepsilon(1+v'v), \\ \eta_{\min} = \varepsilon'\varepsilon(1-v'v). \end{cases} \tag{20}$$

It should also be noted from Eq. (17) that, for a muon emission angle in the pion rest frame given by $\cos\theta = \boldsymbol{v}_I\cdot\boldsymbol{u}_I = -v/u$, the quantity $(\xi\cdot\boldsymbol{v}_I')$ vanishes, so that the muon polarization in the laboratory is transverse. The muon emission angle in the pion rest frame given by $\cos\theta = -v/u$ corresponds to a muon emission angle in the laboratory $\theta' = \cos^{-1}(\boldsymbol{v}_I'\cdot\boldsymbol{u}_I')$ determined by:

$$\operatorname{tg}\theta' = \eta^{-1}\frac{\sin\theta}{\cos\theta+u/v} = \eta^{-1}\frac{\sqrt{1-v^2/u^2}}{u/v-v/u}, \tag{21}$$

---

[6] It is worth mentioning that if one considers throughout, instead of the muon, the neutrino accompanying the muon, then $v'=1$, $v=1$, $\varepsilon = 1/\sqrt{1-(v')^2} = \infty$, $\varepsilon = 1/\sqrt{1-v^2} = \infty$. In this case Eqs. (18), (15), give: $(\xi\cdot\boldsymbol{v}_I') = |\xi| = (\xi\cdot\boldsymbol{v}_I)$ so that the neutrino longitudinal polarization is the same in the laboratory frame as in the pion rest frame.

the first equality in Eq. (21) being a consequence of Eq. (8) (7). As pointed out by Dr. A. ORKIN-LECOURTOIS, the value of θ' given by the second equality of Eq. (21) is also the maximum possible θ' for given u.

The partial longitudinal polarization of the muon « beam » of given velocity $\boldsymbol{v}'$ is now most easily obtained by averaging the laboratory frame value of the longitudinal polarization of a muon of given $\boldsymbol{v}'$, $(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')$, over the parent pion energy spectrum — in this average $(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')$ is considered a function of the parent pion energy as in Eq. (18). Taking a power law for this spectrum:

$$N(\eta)\,\mathrm{d}\eta = (\mathrm{const})\,\eta^{-\gamma}\,\mathrm{d}\eta\,, \tag{22}$$

and using Eq. (18), we have:

$$\left\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')_{\mathrm{Aver.}} = |\boldsymbol{\xi}|\frac{\int_{\eta_{\min}}^{\eta_{\max}}\left[(1/v'v)(1-\eta/\varepsilon'\varepsilon)\right]N(\eta)\,\mathrm{d}\eta}{\int_{\eta_{\min}}^{\eta_{\max}}N(\eta)\,\mathrm{d}\eta}\right. . \tag{23}$$

For a « flat » pion energy spectrum — $\gamma=0$ — Eqs. (23), (20), give $\left\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')\right\}_{\mathrm{Aver.}}=0$; this result is anticipated on the basis of the discussion in the introduction. In the case $\gamma>2$, evaluation of the integral in Eq. (23) with the limits of Eq. (20), yields:

$$\left\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')\right\}_{\mathrm{Aver.}} = |\boldsymbol{\xi}|\left(\frac{1}{v'v}\right)\left\{1-\frac{\gamma-1}{\gamma-2}\,(1-(v'v)^2)\,\frac{(1+v'v)^{\gamma-2}-(1-v'v)^{\gamma-2}}{(1+v'v)^{\gamma-1}-(1-v'v)^{\gamma-1}}\right\} = \tag{24}$$

$$= |\boldsymbol{\xi}|\left\{\frac{1}{3}\gamma(v'v) + \text{terms in } (v'v)^3 + \ldots\right\},$$

so that, again as anticipated on the basis of the discussion in the introduction, the partial longitudinal polarization of the muon « beam » increases with increasing γ and with increasing $(v'v)$. Evidently Eq. (24) applies with appropriate changes of nomenclature to any two-body decay which is isotropic in the parent particle's rest frame and in which the decay product is longitudinally polarized in this frame.

We now apply the general Eq. (24) to various special situations: as seen in the above derivation of Eq. (24) in every case the muons constitute a « beam » of given velocity $\boldsymbol{v}'$.

(7) We may note that by means of this first equality of Eq. (21), and, of Eq. (17), $(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')$ can be expressed as a function of θ' and u, giving the muon longiutdinal polarization in the laboratory in terms of the muon emission angle in the laboratory and the pion velocity.

2·1. $\pi \to \mu$ *decay from stopped pions in emulsions or bubble chambers.* – Here $N(\eta) \approx (\text{const})\,\delta(\eta - 1)$ which can be approximated by a power law spectrum with a very large γ; also $\eta \approx 1$ implies $v' \approx v$. Eq. (24) then gives:

$$\left\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')\right\}_{\text{Aver.}} \approx |\boldsymbol{\xi}|\left(\frac{1}{v^2}\right)\left\{1 - \frac{1-v^4}{1+v^2}\right\} = |\boldsymbol{\xi}|, \tag{25}$$

so that the muon longitudinal polarization is as complete in the laboratory frame as in the pion rest frame. This result is, of course, expected since in the present circumstances the laboratory frame and the pion rest frame essentially coincide.

2·2. $\pi \to \mu$ *decay in flight of cyclotron produced pions.* – Here $\left\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')\right\}_{\text{Aver.}}$ is empirically known to be fairly close to unity. Also $v = 0.27$, $\varepsilon = 1.04$ and $v' \approx 0.9$, $\varepsilon' \approx 2.3$ so that $(v'v) \approx \frac{1}{4}$; $|\boldsymbol{\xi}|$ can be assumed ≈ 1. Let us first suppose that the geometry of the experiment is such as to permit detection of muons with *all* values of θ (muon emission angle in pion rest frame) between 0 and π, or correspondingly, to permit detection of muons emitted by pions with *all* values of η (parent pion energy) between the $\eta_{\min}$ and $\eta_{\max}$ of Eq. (20). Then the $\left\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')\right\}_{\text{Aver.}}$ calculated in Eq. (24) is relevant for comparison with experiment, and a value of $\gamma \approx 15$ is required to explain the empirical value of, say, 0.8 for $\left\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')\right\}_{\text{Aver.}}$. On the other hand, one may suppose that the geometry of the experiment is such as to permit detection of muons with values of θ between 0 and $\Delta\theta \ll \pi$, *only*, or correspondingly, to permit detection of muons emitted by pions with values of η between $\eta_{\min}$ and $\eta_{\min} + \Delta\eta \ll \eta_{\max}$, *only*. The relevant quantity for the determination of the partial longitudinal polarization of the muon « beam » is then, analogously to Eq. (23),

$$\left\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')\right\}_{\text{Aver.}} = |\boldsymbol{\xi}|\,\frac{\int_{\eta_{\min}}^{\eta_{\min}+\Delta\eta}\left[(1/v'v)(1-\eta/\varepsilon'\varepsilon)\right]N(\eta)\,\mathrm{d}\eta}{\int_{\eta_{\min}}^{\eta_{\min}+\Delta\eta}N(\eta)\,\mathrm{d}\eta}, \tag{26a}$$

so that, to first order in $\Delta\eta$:

$$\left\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')\right\}_{\text{Aver.}} \approx |\boldsymbol{\xi}|\left(1 - \frac{\Delta\eta}{2\varepsilon'\varepsilon v'v}\right). \tag{26b}$$

Use of Eq. (20) for $\eta_{\min}$ and comparison of Eq. (26*b*) with the empirical value of ≈ 0.8 for $\left\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')\right\}_{\text{Aver.}}$ then yields, with the above numerical values for v, ε, v', ε', $|\boldsymbol{\xi}|$,

$$\eta_{\min} \approx 1.8; \qquad \Delta\eta \approx 0.23\,, \tag{27}$$

so that, in this illustration, the experiment detects muons emitted by parent pions with kinetic energies from about 110 MeV to about 140 MeV.

2·3. $\pi \to \mu$ *decay in flight of cosmic ray pions.* – In this case $\gamma \approx 2.5$ ([8]) and $v' \approx 1$ while $v = 0.27$ — thus $(v'v) \approx 0.27$. Assuming again that $|\boldsymbol{\xi}| \approx 1$, we have from Eq. (24):

$$\{(\boldsymbol{\xi} \cdot \boldsymbol{v}'_I)\}_{\text{Aver.}} \approx 0.23\,|\boldsymbol{\xi}| \approx 0.23\,. \tag{28}$$

2·4. $\mathrm{K} \to \mu$ *decay in flight of cyclotron produced or cosmic-ray kayons.* – We assume that $|\boldsymbol{\xi}| \approx 1$ for muons originating in kayon decay — this assumption is supported by the fact that $\mu \to \mathrm{e}$ decay of muons from cyclotron-produced kayons is characterized by about as great an electron momentum-direction asymmetry ([9]) as $\mu \to \mathrm{e}$ decay from cyclotron-produced pions. Further, v, which is here the muon velocity in the kayon rest frame, is 0.91 so that $(v'v)$ is ≈ 0.9. Thus, because of this rather large value of $(v'v)$, $\{(\boldsymbol{\xi} \cdot \boldsymbol{v}'_I)\}_{\text{Aver}}$ is seen from Eq. (24) to be close to unity for all $\gamma \geqslant 2.5$, for example:

$$\begin{cases} \{(\boldsymbol{\xi} \cdot \boldsymbol{v}'_I)\}_{\text{Aver.}} \approx 0.85\,|\boldsymbol{\xi}| \approx 0.85\,, & \text{for } \gamma = 2.5\,, \\ \{(\boldsymbol{\xi} \cdot \boldsymbol{v}'_I)\}_{\text{Aver.}} \approx 0.96\,|\boldsymbol{\xi}| \approx 0.96\,, & \text{for } \gamma = 5.0\,: \end{cases} \tag{29}$$

Physically, the range of possible kayon energies for a given muon energy is so large that even for small γ the polarization is near 1.

3. – Experiment.

To search for the expected partial longitudinal polarization of the muon « beam » we have used the arrangement shown in Fig. 1. The Geiger tube coincidence-anticoincidence telescope $AB\bar{C}$ selects muons stopping in the 3.8 cm moderator which is alternatively aluminum (not depolarizing) and salt (almost completely depolarizing). The upward and downward decay electrons are detected by identical plastic scintillators (Nuclear Enterprises

([8]) For estimates of γ from the cosmic ray data, see G. Puppi: *Progress in Cosmic Ray Physics*, **3**, 341 (Amsterdam, 1956).

([9]) C. A. Coombes, B. Cork, W. Falbraith, G. R. Lambertson and W. A. Wenzel: *Phys. Rev.*, **108**, 1348 (1957). In this experiment, the kayons are stopped in a moderator and the longitudinal polarization of the muons emitted in a particular direction is studied. Thus, as in the discussion of Eq. (25), the muon longitudinal polarization is here as complete in the laboratory frame as in the kayon rest frame, the two frames essentially coinciding.

Ltd. Type NE 102) of dimensions 3.8 cm × 18 cm × 28 cm, each viewed through a short lucite light pipe by a single photo-multiplier tube (Dumont Type 6364).

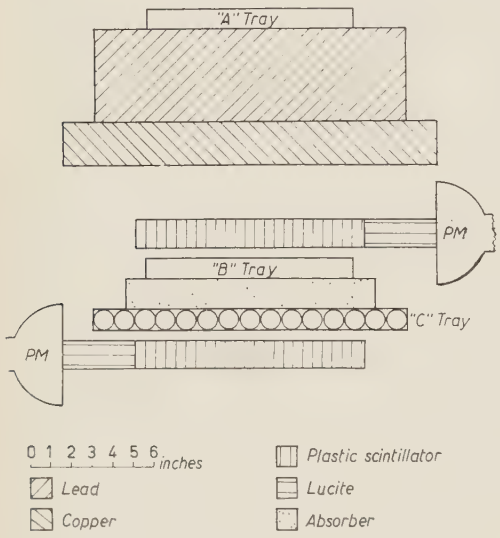

Fig. 1. – Arrangement of counters, absorber, and filtering materials.

The photo-multiplier pulses are fed to the balanced inputs of a Tektronix Type 53G differential pre-amplifier, so that a pulse from the upper scintillator produces a downward deflection of the spot on the scope face of a Tektronix Type 531 cathode-ray oscilloscope while one from the lower scintillator produces an upward deflection. The horizontal sweep, 10 μs long, is triggered by the $(AB\bar{C})$ pulse, with an overall delay, including estimated average Geiger tube lags, of about 0.9 μs. The face of the cathode-ray tube is photographed continuously on slowly moving film. A sample record is shown in Fig. 2. It provides for each delayed photomultiplier pulse both the height and the time delay from the start of the sweep.

The experiment consists of a comparison of the number of $\mu^+ \to e^+$ disintegrations recorded in the upper and in the lower scintillators. Denoting the partial longitudinal polarization of the muon « beam », $\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')\}_{\text{Aver.}}$, by δ, taking the angular distribution of decay electrons from moderated muons, 100% longitudinally polarized at the instant of decay, to be

$$1 - (a)\cos\psi\,, \tag{30}$$

where a, ψ are as in Eq. (13*b*), and neglecting for the moment depolarization effects, we expect (consistent with Eq. (13*b*)) that the angular distribution of decay electrons in the cosmic-ray case will be:

$$1 - (\delta)(a)\cos\psi\,. \tag{31}$$

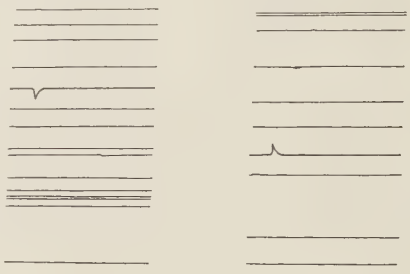

Fig. 2. – Sample film record. The film moves continuously; the oscilloscope sweep, 10 μs long, is triggered by each (ABC) event. Most of the sweeps show no decay electron. On the left, there is a trace showing a pulse from the upper scintillator delayed 1.6 μs from the start of the sweep. On the right there is a trace showing a pulse from the lower scintillator delayed 1.3 μs.

To allow for possible muon depolarization in the atmosphere and within the moderator, the angular distribution of Eq. (31) must be replaced by

$$1 - (\delta k_{\text{atm}}^{(+)} k_{\text{mod}}^{(+)} a) \cos\psi \,, \tag{32}$$

where $k_{\text{atm}}^{(\pm)}$, $k_{\text{mod}}^{(\pm)}$ are the appropriate $\mu^{\pm}$ depolarization coefficients in the indicated media. The angular distribution of Eq. (32) is the proper form for analysis of the experimental results.

If all the muons entered the telescope vertically and the scintillators responded only to vertical electrons, the ratio of the numbers of decay electron pulses in the upper and lower scintillators would be simply

$$\frac{1 + (\delta k_{\text{atm}}^{(+)} k_{\text{mod}}^{(+)} a)}{1 - (\delta k_{\text{atm}}^{(+)} k_{\text{mod}}^{(+)} a)} \,. \tag{33}$$

The finite aperture of the telescope and the finite solid angles of the scintillators reduce this ratio. A rough estimate gives as the corresponding expected upper to lower ratio, R:

$$R = \frac{1 + 0.70(\delta k_{\text{atm}}^{(+)} k_{\text{mod}}^{(+)} a)}{1 - 0.70(\delta k_{\text{atm}}^{(+)} k_{\text{mod}}^{(+)} a)} \,. \tag{34}$$

With the filter of 129 g/cm² Pb followed by 40 g/cm² Cu the stopping muons have, on entry into the telescope, momenta between 337 MeV/c and 358 MeV/c (Al moderator) or 353 MeV/c (NaCl moderator). The analysis of Sands [10] shows that the large majority of these muons result from decay in flight near the top of the atmosphere, so that the assumptions involved in the derivation of Eq. (24) apply and we may use $\delta \approx 0.23$ (Eq. (28)). Theoretical estimates due to HAYAKAWA [2] indicate that $k_{\text{atm}}^{(+)} \approx 1$. In aluminum $k_{\text{mod}}^{(+)} \approx 1$ as the decay electron emission asymmetry observed there in cyclotron experiments is maximal [11]. The value of a, integrated over the whole decay electron energy spectrum, is $\frac{1}{3}$ on a two-component neutrino theory with negative helicity, neutrinos having interfermion vector and axial vector coupling constants equal except for sign [12] — this value of a is consistent with the results of cyclotron experiments in substances with $k_{\text{mod}}^{(+)} \approx 1$ [13,11] if it is assumed that $\{(\boldsymbol{\xi} \cdot \boldsymbol{v}_I')\}_{\text{Aver.}}$

[10] M. SANDS: *Phys. Rev.*, **77**, 180 (1950).

[11] S. C. WRIGHT: *Proceedings of the* 1957 *Rochester Conference*, Chap. VII (New York, 1957).

[12] T. D. LEE and C. N. YANG: *Phys. Rev.*, **105**, 1671 (1957); L. LANDAU: *Nuclear Physics*, **3**, 127 (1957); A. SALAM: *Nuovo Cimento*, **5**, 299 (1957).

[13] D. BERLEY, T. COFFIN, R. L. GARWIN, L. M. LEDERMAN and M. WEINRICH: *Phys. Rev.*, **106**, 835 (1957).

is ≈ 0.8 in the cyclotron case. Now, while our apparatus does not detect low energy decay electrons, their total number is relatively small and it is a sufficiently good approximation for us to use the above value of $a = \frac{1}{3}$ which is appropriate to the whole decay electron energy spectrum ([14]); using also the aforementioned values of δ, $k_{\text{atm}}^{(+)}$, $k_{\text{mod}}^{(+)}$, we expect in Al an upper to lower ratio, R (Eq. (34)), of about 1.11. With NaCl as moderator, $k_{\text{mod}}^{(+)} \approx 0.15$

$$\left([\{(\boldsymbol{\xi}\cdot\boldsymbol{v}_I')\}_{\text{Aver.}}]_{\text{cycl.}}\cdot[k_{\text{mod}}^{(+)}]_{\text{NaCl}}\cdot(a) \approx (0.8)\cdot[k_{\text{mod}}^{(+)}]_{\text{NaCl}}\cdot(\tfrac{1}{3}) \approx 0.04\ ([11])\right),$$

and the upper to lower ratio R (Eq. (34)) is expected to be about 1.02. These expected values for the ratio R have been entered in the fifth column of Table III.

The main difficulty associated with the present experiment involves ensuring equal detection efficiencies for the upward and downward decay electrons. The situation is unsymmetrical in that the stopping muon traverses the upper but not the lower scintillator (this fact prevented us from using the Geiger tubes in the B and C trays to detect decay electrons, because of the insensitivity of the tube traversed by the muon). It is essential that the prompt — and usually larger — scintillator pulse made by the stopping muon not overload the amplifier or reduce in any way the response to a subsequent decay electron pulse. The Tektronix amplifier has been found to be eminently satisfactory in this respect.

However, even with dead-time effects eliminated, an inequality in the over-all gains of the upper and lower decay electron detectors would produce a spurious asymmetry. In effect, the pulse heights recorded are necessarily limited to those exceeding a certain minimum value on the film of the order of the trace width. Now the decay electron pulses are mostly well above this cut-off, but their distribution is continuous, with a tail going below it. Similarly, the noise pulses are mostly below the cut-off, but their spectrum has a tail extending above it. If the overall gains of the upper and lower detectors are not equal, the *fraction* of decay electrons as well as the *fraction* of noise pulses recorded in the two detectors will be different, and a spurious asymmetry will be introduced.

To detect any residual asymmetry, measurements were made alternately with aluminum and salt as absorbers. By taking the ratio of the aluminum up to down ratio to that for salt, instrumental asymmetries are cancelled out. Alternatively, if the up to down ratio for salt is equal to 1.02 within the sta-

([14]) Thus, assuming that our apparatus only detects decay electrons with energy $\geqslant 10$ MeV entails use of a value of a only 3% greater than $\frac{1}{3}$. T. KINOSHITA and A. SIRLIN: *Phys. Rev.*, **107**, 593 (1957), Eq. (3.12).

tistical error, the upper and lower detector efficiencies are equal to within that error.

4. – Results.

The film was scanned for pulses starting on the sweep and greater in height than 0.3 mm, and their position and height were recorded. To convert position to time delay, the sweep was calibrated periodically by photographing the 5 MHz sine wave of the standard frequency transmission from the National Bureau of Standards (WWV). The delay in starting the sweep relative to the time of arrival of the muon was not known precisely. Using fast-rising, artificial pulses to simulate the Geiger signals the delay inherent in the sweep triggering circuitry was found to be (0.7 ± 0.1) μs, the uncertainty corresponding to play in the trigger sensitivity control. With real pulses an additional fluctuating delay caused by the Geiger tubes of about 0.2 μs is introduced. Thus we assume an over-all delay between the occurrence of an $AB\bar{C}$ event and the start of the sweep of (0.9 ± 0.2) μs.

The pulse height distribution exhibits a broad maximum in the region $(1 \div 20)$ mm (as measured on the film) with a valley between 0.3 and 1.0 mm, presumably where the tail of the noise distribution meets the rise of the decay electron distribution. In our analysis of the data we have used all pulses greater than 0.3 mm, relying on the analysis of the time distribution to separate out the background pulses.

The time distributions are shown in Table I. In each moderator there

TABLE I. – *Time distribution of pulses.*

		Microseconds from front of sweep										
		0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10	Total
	$i =$	0	1	2	3	4	5	6	7	8	9	
Al	Upper	291	155	100	61	43	44	17	19	15	9	754
	Lower	255	135	97	53	37	22	14	14	14	9	650
NaCl	Upper	349	163	106	80	55	47	34	17	17	2	870
	Lower	326	166	118	69	53	28	24	20	6	2	812

was, of course, the same running time for the upper and lower electron detectors. The total running times for the two moderators were $4.5 \cdot 10^2$ and $9.6 \cdot 10^2$ h for Al and NaCl respectively.

The differential time distribution is expected to be the sum of three terms. The predominant one, resulting from the decay electrons from positive muon decay, is a decaying exponential with a decay constant, λ_0, of $((2.21 \pm 0.02)\ \mu s)^{-1}$. The second term, significant only in the early channels, results from that fraction of the negative muons which do not annihilate with nuclei. This term is a decaying exponential with a decay constant, λ_-, which depends on the moderator material and is larger than λ_0. In NaCl, this second term itself consists of two components, one from the decays of negative muons trapped in Bohr orbits on Na, the other from decays in Bohr orbits on Cl. Recent measurements at the Chicago synchro-cyclotron ([15]) show equal frequencies of trapping on these two elements with $\lambda_- = (0.829 \pm 0.021)\ \mu s^{-1}$ and $(1.83 \pm 0.09)\ \mu s^{-1}$ for Na and Cl respectively ([16]). For Al we also use the Chicago group's value of $\lambda_- = (1.238 \pm 0.044)\ \mu s^{-1}$. These λ_- values for Na and Al are in excellent agreement with other measurements ([17]).

The third term, a time-independent background, is due to pulses not related to the $AB\bar{C}$ event that triggered the sweep. The minimum ionizing cosmic ray pulses, due to their infrequency, make a negligible contribution to this background and it is apparently due to small pulses from local radioactivity. This is suggested by a time distribution analysis of the small, less than 1 mm, pulses which shows them to be almost uniformly distributed.

We need therefore to extract from the composite time distribution the strength of the positive muon 2.2 μs component. To avoid bias in visual curve fitting and to permit a meaningful statement of experimental error, we have used a least squares procedure.

Labelling the time channels by the integer $i = 0, 1, 2, \ldots, 9$; and denoting the number of pulses in the i-th channel by y_i, we expect the mean value of y_i over many repetitions of the experiment, $\hat{y}_i$, to be given by

$$\hat{y}_i = A\alpha_i + C\,, \tag{35a}$$

with

$$A = N_{\mu^+} \exp\left[-\lambda_0 t_0\right](1 - \exp\left[-\lambda_0 \Delta\right])G(+)\,, \tag{35b}$$

$$\alpha_i = \exp\left[-\lambda_0 i\Delta\right]\left\{1 + \frac{45}{55}\exp\left[-(\lambda_- - \lambda_0)t_0\right]\frac{\lambda_0}{\lambda_-}\cdot \right. \tag{35c}$$

$$\left. \cdot\frac{1 - \exp\left[-\lambda_-\Delta\right]}{1 - \exp\left[-\lambda_0\Delta\right]}\frac{G(-)}{G(+)}\exp\left[-(\lambda_- - \lambda_0)i\Delta\right]\right\}.$$

([15]) V. L. TELEGDI: *Proceedings of the Padua-Venice Conference* (1957) Mimeographed edition, p. IV-62.

([16]) J. C. SENS, R. A. SWANSON, V. L. TELEGDI and D. D. YOVANOVITCH: *Phys. Rev.*, **107**, 1464 (1957).

([17]) For Na, D. R. JONES: *Phys. Rev.*, **105**, 1591 (1957). For Al, L. M. LEDERMAN and M. WEINRICH: *Proceedings of the CERN Conference*, Vol. II (Geneva, 1956); R. D. SARD and M. F. CROUCH: *Progress in Cosmic Ray Physics*, **2**, 12 (Amsterdam, 1954).

Here C is the background per channel; N_{μ^+} is the number of positive muons stopped in the absorber; $G(+,)$ is a factor giving the average detection efficiency for decay electrons from positive and negative muons respectively — clearly, $G(-,)$ involves the spatial distribution of the decay electrons; t_0 (= 0.9 ± 0.2 μs) is the delay of a pulse that appears at the start of the sweep, and Δt (= 1.00 μs) is the channel width used; the negative to positive ratio N_{μ^-}/N_{μ^+} is taken to be 45/55 [18]. We wish to find from our data the ratio R of Eq. (34) — R, in turn, is equal to the ratio: $G(+,$ upper detector$)/G(-,$ lower detector$)$. From Eq. (35*b*) this last ratio equals A(upper)/A(lower); thus our least-squares procedure which will give us the value of A in each case will determine R experimentally.

The function α_i involves the ratios $G(-,$ upper$)/G(+,$ upper$)$ or $G(-,$ lower$)/G(+,$ lower$)$ but not in a sensitive way. In our computation of α_i we therefore assume values for these ratios expected from the discussion after Eq. (34) and from the fact that $I_{\mu^-} \sim 0.6\, I_{\mu^+}$, *viz.*: 0.95 for (Al, upper) 1.05 for (Al, lower), 1.00 for (NaCl, upper or lower). For NaCl, the second term in curly brackets in Eq. (35*c*) is split into two terms of the same form, each with the factor $\frac{1}{2}$, and the λ_0 referring to Na and Cl respectively. The so computed values of α_i corresponding to the channels of Table I are given in Table II. The error in A, induced by the approximations employed in the evaluation of the α_i, is discussed below.

TABLE II. – *Values of the functions* α_i.

	i = Channel number	0	1	2	3	4	5	6	7	8	9
Al	Upper	1.2735	0.7150	0.4271	0.2636	0.1652	0.1044	0.06615	0.04201	0.02669	0.01696
	Lower	1.3023	0.7234	0.4295	0.2643	0.1654	0.1044	0.06617	0.04201	0.02670	0.01696
NaCl	Upper	1.3138	0.7541	0.4528	0.2777	0.1723	0.1077	0.06769	0.04270	0.02701	0.01710
	Lower	1.3138	0.7541	0.4528	0.2777	0.1723	0.1077	0.06769	0.04270	0.02701	0.01710

[18] The λ_0 appearing in the second term in the curly brackets of Eq. (35*c*) is the decay rate of a bound negative muon, while elsewhere λ_0 is the decay rate of a positive muon. We identify these two rates though they are actually somewhat different because of Bohr orbit binding effects; however this difference is small for the small Z that are used in the present experiment (see J. C. SENS, R. A. LUNDY, R. A. SWANSON, V. L. TELEGDI and D. D. YOVANOVITCH: *Bull. Am. Phys. Soc.*, **3**, 198 (1958)). In addition, as pointed out below, any error in λ_0 has no effect in first order on the value of the ratio R with which we are concerned.

The least-squares procedure can now be derived by maximizing as a function of the parameters A and C the likelihood of obtaining the actual result. The width of the likelihood peak then gives a measure of the uncertainty of the estimates of A and C. Assuming Gaussian distributions of y_i about $\hat{y}_i$ with variance $\hat{y}_i$, we have for the logarithm of the likelihood:

$$\log \mathcal{L} = -\sum_{i=0}^{9}\left[\frac{(y_i-\hat{y}_i)^2}{2\hat{y}_i}+\frac{1}{2}\log \hat{y}_i+\frac{1}{2}\log(2\pi)\right]. \tag{36}$$

The maximum likelihood estimates of A and C are found from the simultaneous solutions of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}A}\log\mathcal{L} = \displaystyle\sum_{i=0}^{9}\left(\frac{\mathrm{d}}{\mathrm{d}\hat{y}_i}\log\mathcal{L}\right)\cdot\alpha_i = 0\,, \\ \dfrac{\mathrm{d}}{\mathrm{d}C}\log\mathcal{L} = \displaystyle\sum_{i=0}^{9}\left(\frac{\mathrm{d}}{\mathrm{d}\hat{y}_i}\log\mathcal{L}\right)\cdot 1 = 0\,. \end{cases} \tag{37}$$

The main contribution to $\mathrm{d}/\mathrm{d}\hat{y}_i \log \mathcal{L}$ arises from the numerator of the first term of $\log \mathcal{L}$:

$$\frac{\mathrm{d}}{\mathrm{d}\hat{y}_i}\log\mathcal{L} = \frac{y_i-\hat{y}_i}{\hat{y}_i}\left[1+\frac{1}{2}\,\frac{(y_i-\hat{y}_i)^2-\hat{y}_i}{\hat{y}_i(y_i-\hat{y}_i)}\right] = \frac{y_i-\hat{y}_i}{\hat{y}_i}\left[1+O\left(\frac{1}{\sqrt{\hat{y}_i}}\right)\right], \tag{38}$$

so that in the last expression we may neglect the term $O(1/\sqrt{\hat{y}_i})$ compared to 1 and replace the $\hat{y}_i$ in the denominator by y_i. This is equivalent to regarding the variance of each y_i as known *a priori* and equal to y_i. The Eqs. (37) then become

$$\begin{cases} \displaystyle\sum_{i=0}^{9}\left(\frac{y_i-\hat{y}_i}{y_i}\right)\cdot\alpha_i = 0\,, \\ \displaystyle\sum_{i=0}^{9}\left(\frac{y_i-\hat{y}_i}{y_i}\right) = 0\,, \end{cases} \tag{39}$$

and the solutions for A and C are found from Eqs. (39), (35a) as:

$$A = \frac{a_{22}I_1 - a_{22}I_2}{\mathcal{D}}; \qquad C = \frac{a_{11}I_2 - a_{12}I_1}{\mathcal{D}}\,, \tag{40}$$

where

$$\begin{cases} a_{11} = \displaystyle\sum_{i=0}^{9}\frac{\alpha_i^2}{y_i}\,, & a_{12} = \displaystyle\sum_{i=0}^{9}\frac{\alpha_i}{y_i}\,, & a_{22} = \displaystyle\sum_{i=0}^{9}\frac{1}{y_i}\,, \\ \mathcal{D} = a_{11}a_{22} - a_{12}^2\,, & I_1 = \displaystyle\sum_{i=0}^{9}\alpha_i\,, & I_2 = \displaystyle\sum_{i=0}^{9}1 = 10\,. \end{cases} \tag{41}$$

The values of A and C as computed by these formulas are given in Table III. The number of $\mu^+ \to e^+$ decay pulses recorded is $(1 - \exp[-\lambda_0 \Delta])^{-1} A = 2.74A$; the number of background pulses is $10C$; the remaining pulses are $\mu^- \to e^-$ decays.

TABLE III. – *Results.*

		$A \pm \sigma_A$	$C \pm \sigma_A$	$R = A$ (upper)/A (lower)	
				exper.	theor.
Al	Upper	215 ± 10	8.9 ± 3.8	$1.14 \pm .07$	1.11
	Lower	189 ± 9	5.3 ± 3.0		
NaCl	Upper	254 ± 10	1.1 ± 1.3	$1.03 \pm .06$	1.02
	Lower	247 ± 10	0.4 ± 1.4		

The statistical errors in A and C are specified by the dimensions of the « standard deviation » contour of the likelihood surface in the (A, C) plane [19], *i.e.* that contour for which $\mathcal{L}$ is reduced from its peak value by the factor $\exp[-\frac{1}{2}]$. Stopping with second order terms in the Taylor expansion of $\log \mathcal{L}$ about its peak value, we find the contour to be an ellipse. From its dimensions we obtain for the variance of the estimates of A and C:

$$\sigma_A^2 = \frac{a_{22}}{\mathcal{D}}; \qquad \sigma_C^2 = \frac{a_{11}}{\mathcal{D}}. \tag{42}$$

Here we have continued to treat the variances of the y_i as equal to y_i. The simple relations obtained in Eq. (42) are the same as those which describe how much the maximum likelihood peak shifts when the input data y_i are subjected to « statistical variation »; such a shift is obtained by differentiating Eq. (40) for A and C with respect to the y_i. The values of σ_A and σ_C calculated from Eqs. (42), (41) are shown in Table III.

Systematic errors are contained in the coefficient α_i which is treated as known in the above least-squares analysis. Uncertainties in the quantities λ_0, λ_-, t_0, $G(-)/G(+)$, entering into α_i influence the various terms in Eq. (40) for A. Errors in λ_0, λ_-, t_0, within the α_i affect A (upper) and A (lower) in the same way and so contribute no error in first order to $R = A$ (upper)/A (lower); on the other hand, any error in $G(-,$ upper$)/G(+,$ upper$)$ and $G(-,$ lower$)/$

[19] M. ANNIS, W. CHESTON and H. PRIMAKOFF: *Rev. Mod. Phys.*, **25**, 818 (1953).

$/G(+$, lower) within the α_i will in general contribute a first order error to A (upper)/A (lower). However, numerical calculation shows that if, for example, $G(-)/G(+)$ is taken as 1 both for (Al, upper) and (Al, lower) instead of 0.95 and 1.05 respectively, the result for A (upper)/A (lower) is only decreased by 0.6%. The error consequent upon the choice actually made for $G(-)/G(+)$ is therefore negligible.

The fourth column of Table III shows the final results of the experiment, while the fifth column presents the theoretical expectations. Within the limits of the statistics there is excellent agreement, the decays from aluminum showing the expected preference for backward emission and those from salt exhibiting the expected isotropy. The latter result establishes experimentally the over-all symmetry of the apparatus.

5. – Conclusion.

We have found that the decay electrons from cosmic-ray positive muons brought to rest in aluminum tend to be emitted in the backward direction, the amount of anisotropy corresponding to a partial longitudinal polarization of the muons of approximately 25%. This result agrees within the experimental error with that calculated on the following assumptions:

1) cosmic ray muons result preponderantly from the decay in flight of pions with a differential energy spectrum of the form $\eta^{-\gamma}\mathrm{d}\eta$, with $\gamma = 2.5$;

2) in the rest frame of the parent pion, the muon is emitted isotropically with 100% longitudinal polarization;

3) the relativistic transformation properties of the muon spin are those appropriate to a Dirac particle of spin $\frac{1}{2}$;

4) there is negligible depolarization of the muon as it loses 2 GeV energy in coming down through the atmosphere ([2]);

5) the cyclotron experiments allow calculation of sufficiently accurate values for (*a*): the muon depolarization coefficients in various moderators and (*b*): the coefficient in the angular distribution of decay electrons from moderated muons with 100% longitudinal polarization. The present experiment therefore constitutes partial confirmation of all these assumptions. As the statistical errors are still sizeable, the experiment is being repeated with an arrangement giving considerably higher counting rates.

* * *

We shoud like to thank Mr. J. D. Miller for essential help in making and maintaining the equipment and Messrs. P. Smith and W. Mullin for their assistance in operating the equipment and reducing the data.

RIASSUNTO (*)

Per effetto dello spettro d'energia declinante dei pioni generatori si prevede che le particelle μ dei raggi cosmici di energia e direzione di moto definite abbiano polarizzazione longitudinale parziale. Si elabora la teoria generale della polarizzazione dei μ nei decadimenti $\pi \to \mu$ in volo e la si applica alle esperienze, sia al ciclotrone sia coi raggi cosmici. Si prevede che i μ dei raggi cosmici siano polarizzati circa al 23%. La teoria si applica anche al decadimento $K \to \mu$. Si è eseguita un'esperienza per verificare la previsione fatta pei μ dei raggi cosmici. Gli elettroni di decadimento dei μ arrestati in un assorbitore d'alluminio si rivelano in contatori a scintillazione identici posti simmetricamente sopra e sotto l'assorbitore. Si sono misurati il ritardo e l'ampiezza di ogni impulso osservato e si sono impiegati per separare i decadimenti $\mu^+ \to e^+$ dai contributi dovuti ai $\mu^- \to e^-$ ed al fondo. Il rapporto degli eventi $\mu^+ \to e^+$ rivelati dagli scintillatori superiore e inferiore è 1.14 ± 0.07 contro una previsione teorica di circa 1.11. Per controllare la simmetria dell'apparecchio si è ripetuto l'esperimento con un assorbitore di sale, che, notoriamente, depolarizza quasi completamente i muoni arrestati. Il rapporto degli eventi superiori agli inferiori è in questo caso 1.03 ± 0.06 da confrontare colla previsione teorica di circa 1.02. Si conclude che le varie ipotesi fatte nella teoria della polarizzazione dei muoni nel decadimento $\pi \to \mu$ in volo sono corrette e che i muoni subiscono una depolarizzazione trascurabile mentre perdono 2 GeV di energia nell'atmosfera.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

Hyperfragment Binding Energies (*).

S. Limentani, P. E. Schlein, P. H. Steinberg and J. H. Roberts

Department of Physics, Northwestern University - Evanston, Ill.

(ricevuto il 3 Giugno 1958)

Summary. — A study of hyperfragments has been made in a 10-inch^3 stack of Ilford G5 emulsion which was exposed to the Berkeley 4.7 GeV/c π^- beam. In area scanning 66 500 interaction stars, 37 hyperfragments (12 mesonic decays — 25 non-mesonic decays) and 61 other secondary events (« GOKS ») with connecting tracks < 15 μm were found. Some fraction of these 61 GOKS (undetermined in this study) represent hyperfragments. Binding energies for 6 of the mesonic hyperfragments have been obtained. Three of these binding energies are for $Z \geqslant 3$.

1. – Introduction.

The answers to many questions [1] concerning Λ^0-N force depend upon a more complete knowledge of the binding energies in the various hyperfragments. In this paper are reported binding energies for several hyperfragments, some of which are rare or hitherto unknown.

2. – Exposure, processing and scanning.

Forty-eight 3 in. × 3 in. × 600 μm Ilford G-5 pellicles were exposed to the 4.7 GeV/c π^- channel at the Bevatron. The mounted unprocessed pellicles

(*) Work supported by National Science Foundation and U. S. Atomic Energy Commission through Argonne National Laboratory Subcontract.

(1) L. M. Brown and M. Peshkin: *Phys. Rev.*, **107**, 272 (1957); R. H. Dalitz: *Proc. of the Phys. Soc.* (London, 1957); 8 Lectures on *The Strong Interactions of Strange Particles* (Brookhaven, 1957).

were allowed to dry for 4 days before processing, as an attempt to reduce the bubbling during processing [2]. The result was: Twenty-seven visible bubbles of diameter ~ 6 mm over the entire 3 square feet of emulsion surface.

The pellicles were area scanned under 120× magnification, and all visible prongs of the primary interactions were followed out until they either stopped, left the pellicle, or interacted. All secondary events were recorded. 37 hyperfragments (12 mesonic decays, 25 non-mesonic decays (*)) and 61 other secondary events (« GOKS ») with connecting tracks < 15 μm were found. Some fraction of these 61 GOKS represent hyperfragments (undetermined in this study). Analysis of only the mesonic decays has been carried through.

3. – Measurements.

After the exposure and before processing, the thickness of each pellicle was measured in 5 places with a calibrated dial gauge operating vertically on a machined flat steel plate. The mean plate thickness for the stack was found to be (613 ± 13) μm, where 13 μm is the r.m.s. deviation in the distribution of mean initial plate thicknesses. The mean r.m.s. deviation of original thicknesses within a given pellicle from the mean initial thickness for that pellicle was $(2.5 \pm 1.0)\%$ (where 1.0% is again the σ of this distribution).

Track lengths with projected lengths > 100 μm, were measured with the use of 4 PAG-FM 1000 dial gauges (+) on the x, y and z motions. Each gauge was calibrated with the 100 μm lines on several Bausch and Lomb stage calibration scales and the calibration curves (typical curve shown in Fig. 1) were

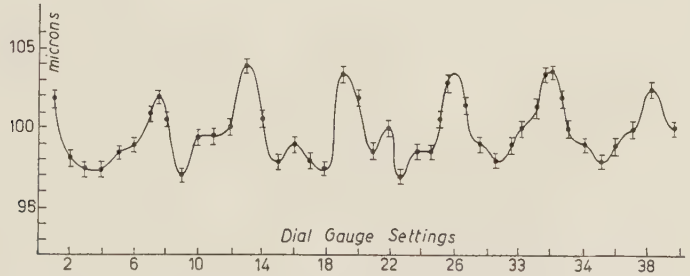

Fig. 1. – Calibration curve of typical dial gauge (PAG-FM 1000) used in linear measurements. The $\sim(600 \div 700)$ μm periodic fluctuations have peak amplitudes of $\pm 4\%$.

(2) E. J. BURGE, J. H. DAVIES, I. J. VAN HEERDEN and D. J. PROWSE: *Nuovo Cimento*, **5**, 1005 (1957).

(*) Essentially the same selection criteria for non-mesonic hyperfragments as described in J. SCHNEPS: *Thesis* (University of Wisconsin) was used in this work.

(+) Manufactured by: A. A. G. FLURY: *Prazisions-Apparatebau* (Grenchen, Switzerland).

included in an IBM 650 program used to calculate the track lengths. The periodic nature (period $\sim 600\ \mu m$) of the non-linearities of these gauges is known to be a general characteristic of them, since other laboratories have found the same effect.

Measurement of dip angles and track lengths $< 100\ \mu m$ is accomplished with the use of the Z dial gauge and a calibrated eyepiece scale (or filar micrometer eyepiece). A linear region of the dial gauge is used for measurement of final thicknesses and Δz of the track in question.

4. – Results.

Of the 12 mesonic decays found, 6 furnish binding energies; 2 of the pions leave the stack, 3 interact in flight; one event has a non-measurable, and thus unidentifiable, recoil. The identification and binding energies (*) furnished are:

Event	Type	Decay Mode	Binding Energy (MeV)
119	$^9Li_\Lambda$	$\rightarrow 2\,^4He + n + \pi^-$	6.57 ± 0.71
156	$^8Be_\Lambda$	$\rightarrow\ ^7Be + {}^1H + \pi^-$	4.98 ± 0.51
194	$^7Li_\Lambda$	$\rightarrow\ ^6Li + {}^1H + \pi^-$	5.27 ± 0.40
142	$^5He_\Lambda$	$\rightarrow\ ^4He + {}^1H + \pi^-$	2.15 ± 0.53
167			2.02 ± 0.45
189	$^4He_\Lambda$	$\rightarrow\ ^3H + {}^1H + \pi^0$	3.8 ± 2.3

A detailed analysis of the events follows, after which we discuss to some extent the evaluation of errors and the calibrations made in order to reduce them. The errors quoted for the binding energies do not include the systematic errors in Q_Λ and in the range-energy relation.

Event 119 - $^9Li_\Lambda$: This event, shown in Fig. 2, originated in a $11+4\pi^-_{star}$ at a production angle of 117.0° from beam direction, and consists of two saturated prongs (24.1 μm and 39.1 μm) and a steep (79.2°) pion of range 17.74 mm. The connecting track shows thin-down and indicates probable $Z = 3$. The plane of decay is almost perfectly normal to the emulsion plane and it can be noticed visually that the emulsion plane projections of the three

(*) As in W. E. Slater: *Thesis* (University of Chicago), we assume throughout our present work, the value $Q_\Lambda = (37.22 \pm 0.2)$ MeV which is a weighted average of the U.C.R.L. value of (37.45 ± 0.17) MeV and the Bristol value of (36.9 ± 0.2) MeV.

prongs all lie to one side of a straight line drawn through the decay point. A neutron(s) must therefore be involved. On the other hand, measurements indicate a very small degree of non-coplanarity, suggesting a very low mo-

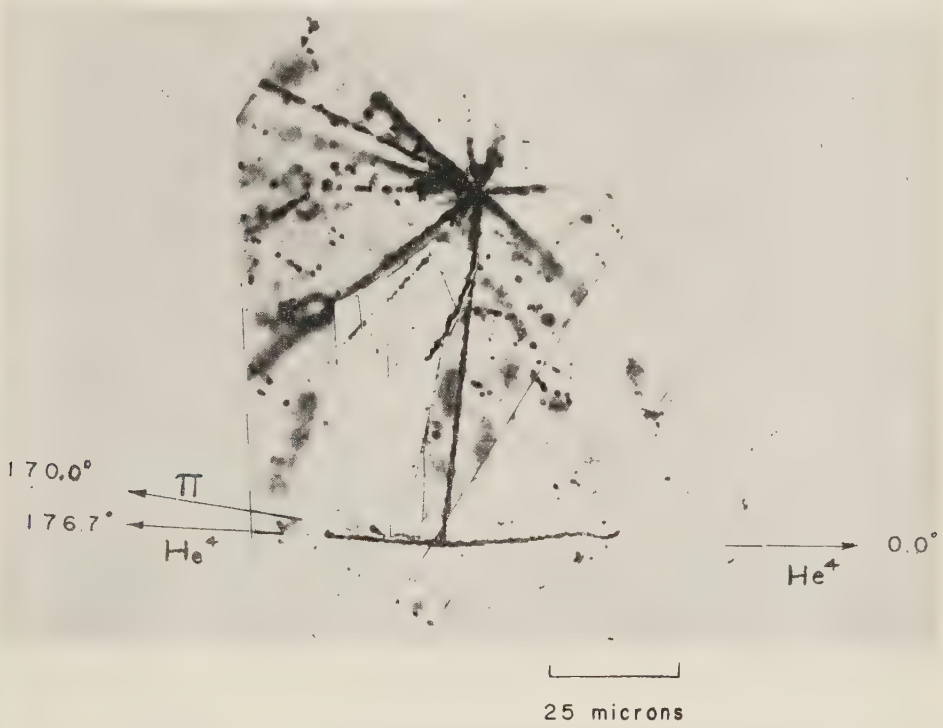

Fig. 2. – Photograph of Event 119 ($^9Li_\Lambda$). Several grains of the sharply rising pion (79.2°) are shown in the picture. The small degree of non-coplanarity can be seen.

mentum for the neutron(s). With the use of the IBM 650 hyperfragment analysis program ([3]), the momentum unbalance of these three prongs for all combinations of charge and mass assumptions from $Z=1$ to $Z=3$ (higher Z assumptions give results completely inconsistent with all existing hyperfragment data) has been calculated and attributed to a single neutron. B_Λ's are then calculated. It is found that for the assumption:

$$^9Li_\Lambda \rightarrow 2\,^4He + \pi^- + n$$

([3]) F. INMAN: *Thesis* UCRL (Report n. 3815); W. H. BARKAS: *Proc. of the 7th Annual Rochester Conference* (1957), p. VIII-11.

the visible Q is 45.73 ± 0.58 MeV which leads to $B_\Lambda = 6.57 \pm 0.71$ MeV. All other prong, charge, and mass assumptions lead to B_Λ's which are highly negative. Assuming one neutron in the final state, the next closest binding energy ((-11.5 ± 0.8) MeV) is furnished by the assumption

$$^8\mathrm{Li}_\Lambda \rightarrow {}^3\mathrm{He} + {}^4\mathrm{He} + \mathrm{n} + \pi^- .$$

The residual momentum for the assumption

$$^8\mathrm{Li}_\Lambda \rightarrow 2\,{}^4\mathrm{He} + \pi^-$$

is (19.9 ± 13.5) MeV/c with a binding energy of (8.82 ± 0.60) MeV. The aforementioned non-coplanarity as well as the high residual momentum and binding energy for the $^8\mathrm{Li}_\Lambda$ assumption make the identification as $^9\mathrm{Li}_\Lambda$ more probable. The binding energy for $^9\mathrm{Li}_\Lambda$ is found to fit on the existing B_Λ versus A curve ([4]).

Events 156, 194, 167, 142: In these three body coplanar events, the π^--proton unbalanced momentum and the measured recoil range are used to identify the recoil. The recoil energy determined directly from the residual π^--proton momentum is then used to calculate the B_Λ's. The normalized momentum-range curves ([4]) derived from Wilkin's report ([5]) guide in the recoil identification:

Event	Recoil Identification	Recoil Range	Recoil Momentum
156	^{7}Be	2.3 ± 0.45	126.6 ± 3.8
194	^{6}Li	6.7 ± 1.0	191.5 ± 2.0
167	^{4}He	19.0 ± 0.6	188.9 ± 5.5
142	^{4}He	8.8 ± 1.0	143.5 ± 2.5

Event 189 - $^4\mathrm{He}_\Lambda \rightarrow {}^3\mathrm{H} + {}^1\mathrm{H} + \pi^0$: Identification of this event is the result of a study of 3 two prong stars which were labeled during scanning as « possible π^0 decays. » In calculating the B_Λ, the residual momentum of the two prongs for all possible combinations of charge and mass assumptions from $Z = 1$ to $Z = 3$ was attributed to a π^0 and the B_Λ calculated therefrom. This

([4]) V. L. TELEGDI: *Proc. of the 7th Annual Rochester Conference* (1957), p. VIII-9-10, on world survey of hyperfragments: R. LEVI-SETTI, W. E. SLATER and V. L. TELEGDI.

( ) J. J. WILKINS: R-E *Relations for Ilford Nuclear Emulsions* (Harwell Report, 1951).

led to unreasonable B_Λ's in all cases but this one. The residual momentum of (101.9 ± 3.8) MeV/c is very close to the π^0 momentum in the free Λ^0 decay. This plus the consistent B_Λ is a fairly good indication of the validity of this identification.

5. – Discussion.

(*a*) The largest contribution thus far to the errors of Λ^0 binding energies of individual mesonic hyperfragments is the 3% range straggling uncertainty of the pion. The contribution of this random error decreases for large n as $1/\sqrt{n}$ when the mean B_Λ of n events of the same species is computed. With the large numbers of mesonic hyperfragments that are currently reported in the literature ([4,6,7]), attempts must be made to reduce the orders of magnitude of the systematic errors well below this value.

The two most important contributing systematic errors are the uncertainty of the Q_Λ of the Λ^0 decay and the uncertainty of the latest BARKAS *et al.* Range-energy relation ($< 1\%$ in energy) ([8]). (The uncertainty in Q_Λ should soon be reduced as the results of several emulsion groups now working with stopping K^- exposures made at the Bevatron in December 1957 are published).

To normalize the Barkas range-energy curve to a given emulsion stack, it is essential that the emulsion density at time of the exposure be determined. Thus the necessity for a good μ length calibration point from a large number of π-μ decays found in the stack. This is particularly important when several binding energies are quoted from the same stack, for then an uncertainty in emulsion density is no longer random, but appears as a systematic error in all the quoted binding energies. With an accepted σ in the μ length distribution from π-μ decays of 30 μm ([9]) and 70 such tracks measured in our stack, we get an error in the mean of 3.0 μm (0.5%) leading to a 0.5% uncertainty in the true decay prong ranges, or ~ 0.08 MeV energy uncertainty from this source.

Other measurement errors random to each event are due to uncertainties in the localized original thicknesses of the pellicles involved. This error appears as a 2.5% r.m.s. error in the shrinkage factor distribution over a typical 600 μm G-5 Ilford pellicle, and affects both dip angle and range determinations. The

---

([5]) J. SCHNEPS, W. F. FRY and M. S. SWAMI: *Phys. Rev.*, **106**, 1062 (1957).

([7]) R. G. AMMAR, R. LEVI SETTI, S. LIMENTANI, J. H. ROBERTS, P. E. SCHLEIN, W. E. SLATER, P. H. STEINBERG: *Bull. Amer. Phys. Soc.* II, **3**, 175 (1958).

([8]) W. H. BARKAS, P. H. BARRETT, P. CÜER, H. HECKMAN, F. M. SMITH and H. K. TICHO: UCRL Report n. 3768, 9.

([9]) W. F. FRY and G. R. WHITE: *Phys. Rev.*, **90**, 207 (1953).

percent contribution to the range uncertainty is then seen to equal $(\sin^2\lambda/\sqrt{n})\mathrm{d}S/S$ (where λ is the dip angle (latitude) of the track involved, n is the number of pellicles through which it passes, and $\mathrm{d}S/S$ is the r.m.s. deviation of the shrinkage factor). For the long meson tracks, this contribution is inconsequential, but for shorter steep prongs, the contribution can become significant. Other random errors in angle and range measurements are, in general, still overshadowed and need no special consideration at this point.

(*b*) The possible existence of real intermediate states in the decay of the ${}^9\mathrm{Li}_\Lambda$ has been considered. Under favorable circumstances this could lead to a determination of the spin of the hyperfragment and information on the spin dependence of the Λ^0-nucleon force. The following decay schemes were considered possible and subjected to kinematic analysis:

(*a*) ${}^9\mathrm{Li}_\Lambda \to {}^9\mathrm{Be}^* + \pi^-$, ${}^9\mathrm{Be}^* \to 2\,{}^4\mathrm{He} + \mathrm{n}$

(*b*) ${}^9\mathrm{Li}_\Lambda \to \mathrm{n} + {}^8\mathrm{Be}^* + \pi^-$, ${}^8\mathrm{Be}^* \to 2\,{}^4\mathrm{He}$

(*c*) ${}^9\mathrm{Li}_\Lambda \to {}^4\mathrm{He} + {}^5\mathrm{He}^* + \pi^-$, ${}^5\mathrm{He}^* \to {}^4\mathrm{He} + \mathrm{n}$.

The kinetic energy in the center-of-mass system of the final state particles was determined in each case. This should give the excitation level of the supposed existent excited nucleus. The resultant «levels» are compared with known levels ([10]); coincidence between a calculated level and a known level then tells us that the hyperfragment could have decayed initially into a state containing this excited nucleus.

	Calculated level	Neighboring known levels
${}^8\mathrm{Be}^*$	12.14 MeV	10.8, 14.7 MeV
${}^9\mathrm{Be}^*$	14.40 MeV	11.3, 17.3 MeV
${}^5\mathrm{He}^*$	0.16 or 0.75 MeV	ground state resonance

([10]) T. LAURITSEN: *Energy levels of light nuclei* (American Institute of Physics Handbook, 1957).

An experimental uncertainty 0.5 MeV on the calculated levels then leads us to the conclusion that this hyperfragment is consistent with having decayed directly to the state:

$$^{9}\mathrm{Li}_{\Lambda} \rightarrow {}^{4}\mathrm{He} + {}^{5}\mathrm{He}^{*} + \pi^{-} ,$$

where ^{5}He* is the ground state resonance level.

* * *

We wish to express our gratitude to Dr. E. J. LOFGREN and his staff at the Bevatron, for their kind co-operation in helping us to obtain our exposure. We would also like to thank Dr. W. BARKAS and Dr. FRED INMAN for making available to us the IBM-650 program for the analysis. We are grateful to Mrs. SHIRLEY LINFIELD, Mrs. JEAN PORTER, Mrs. MILDRED YAHIA and Mr. DAVID AXELROD for their diligent efforts in locating our events, to Mr. PETER KAHN his help in the analysis, and to Dr. D. HARRIS and the staff of the Northwestern University Computing Center for their generous assistance.

Particularly, we wish to express our gratitude to Dr. LAURIE BROWN for his many suggestions and criticisms throughout the course of this work.

RIASSUNTO (*)

In un pacco di 10 in^{3}. di emulsione Ilford G5 esposto al fascio di π^{-} di 4.7 GeV/c di Berkeley è stato eseguito uno studio degli iperframmenti. Con l'esplorazione di superficie si sono trovati 66 500 stelle d'interazione, 37 iperframmenti (12 decadimenti mesonici — 25 decadimenti non mesonici) e 61 altri eventi secondari (« GOKS ») con tracce di collegamento $< 15\,\mu$m. Alcuni di questi 61 GOKS (non determinati quantitativamente nel presente studio) rappresentano iperframmenti. Si sono ottenute le energie di legame per 6 degli iperframmenti mesonici. Tre di queste sono per $Z \geqslant 3$.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

On the Formation of Magneto-Hydrodynamic Shock Waves.

S. Segre

C.N.R.N., Laboratorio Gas Ionizzati
presso Istituto di Fisica dell'Università - Roma

(ricevuto il 10 Giugno 1958)

Summary. — A solution of the time-dependent non-linear equations of magneto-hydrodynamics for the propagation of plane (finite) waves is discussed, for a simple case in which there is a strong analogy with hydrodynamics.

The propagation of waves of infinitesimal amplitude in a conducting fluid in the presence of a magnetic field has been studied extensively by many authors, since the pioneer work of Alfvén. On the other hand, the study of magnetohydrodynamic waves of finite amplitude (shock waves) has attracted attention only quite recently. De Hoffman and Teller [1] and Helfer [2] have considered such shocks, using the analogues of the Rankine-Hugoniot equations. These are relations between the values of the velocity, density and magnetic field, ahead and behind a plane shock for a state of uniform flow, that is, for a stationary shock. Marshall [3] has studied the internal structure of such a stationary shock including in his treatment both viscosity, heat conductivity and finite electrical conductivity. Burgers [4] has studied the internal structure of such a steady shock and the effect of non-uniformity of the magnetic field. These authors do not consider the difficult problem of the build-up of a magneto-hydrodynamic shock, which requires the solution of the time-dependent non-linear equations of motion.

(1) F. de Hoffmann and E. Teller: *Phys. Rev.*, **80**, 692 (1950).
(2) H. L. Helfer: *Astrophys. Journ.*, **117**, 177 (1953).
(3) W. Marshall: *Proc. Roy. Soc.*, A **233**, 367 (1955).
(4) J. M. Burgers: Article in *Magnetohydrodynamics*, edited by R. K. M. Landshoff (Stanford, 1957).

This can be done for the case of plane motion with magnetic field orthogonal to the direction of propagation.

We will use Maxwell's equations

$$\operatorname{curl} \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{i}\,; \qquad \operatorname{curl} \boldsymbol{E} = -\frac{\mu}{c}\frac{\partial \boldsymbol{H}}{\partial t}\,,$$

and the equation $\boldsymbol{i} = \sigma\{\boldsymbol{E} + (\mu/c)\boldsymbol{v}\times\boldsymbol{H}\}$ which defines the electrical conductivity, σ. (We have neglected charge density and displacement current).

If we seek a one-dimensional, unidirectional-flow solution for plane waves (of finite amplitude) propagating in the direction of the x axis, the equation of motion and the equation of continuity will be respectively:

$$\varrho\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{\mu}{c}(\boldsymbol{i}\times\boldsymbol{H})_x - \frac{\partial p}{\partial x} + \varrho\,\delta\,\frac{\partial^2 v_x}{\partial x^2}\,,$$

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho v_x) = 0\,.$$

The coefficient δ is given by $\delta = 4\nu/3 + \mu_v/\varrho + (\gamma-1)k/\varrho c_p$ (cfr. LIGHTHILL, 1956 [5]), for a fluid with perfect gas equation of state and kinematic viscosity ν, bulk viscosity μ_v, thermal conductivity k, specific heat at constant pressure c_p.

The above equations are satisfied by the following solution:

(I)
$$\begin{cases} i_x = 0; & i_z = 0; & v_y = 0; & v_z = 0; \\ H_x = 0; & H_y = 0; & E_z = 0; & E_x = 0\,. \end{cases}$$

(II)
$$\begin{cases} i_y = -\dfrac{c}{4\pi}\dfrac{\partial H_z}{\partial x}\,, \\ E_y = i_y/\sigma + \dfrac{\mu}{c}v_x H_z\,, \end{cases}$$

(III)
$$\begin{cases} (1) \qquad \varrho\dfrac{\mathrm{d}v_x}{\mathrm{d}t} = -\dfrac{\mu}{8\pi}\dfrac{\partial (H_z)^2}{\partial x} - \dfrac{\partial p}{\partial x} + \varrho\,\delta\,\dfrac{\partial^2 v_x}{\partial x^2}\,, \\ (2) \qquad \dfrac{\mathrm{d}\varrho}{\mathrm{d}t} + \varrho\dfrac{\partial v_x}{\partial x} = 0\,, \\ (3) \qquad \dfrac{\mathrm{d}H_z}{\mathrm{d}t} + H_z\dfrac{\partial v_x}{\partial x} = \dfrac{c^2}{4\pi\mu\sigma}\dfrac{\partial^2 H_z}{\partial x^2}\,. \end{cases}$$

(5) Article in *Surveys in Mechanics* edited by G. K. BATCHELOR and R. M. DAVIES (Cambridge, 1956).

If we solve system III for the unknown functions, v_x, ϱ, H_z, then from II we obtain the remaining field quantities.

In the case of infinite electrical conductivity ($1/\sigma = 0$), we obtain from (2) and (3),

$$\frac{H_z}{\varrho} = \text{const}\,, \tag{4}$$

which expresses the fact that the magnetic field is « frozen » in the fluid. Equation (1) is the generalization of Euler's equation by the addition of a magnetic pressure $\mu H^2/8\pi$. This pressure, for $\sigma \to \infty$, is a function of ϱ, just as the hydrodynamic pressure is, in an isentropic transformation. Equation (1) becomes, for $\sigma \to \infty$,

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\frac{\mu H_z^2}{8\pi} + p\right)\frac{1}{\varrho}\frac{\partial \varrho}{\partial x} + \delta\,\frac{\partial^2 v_x}{\partial x^2}; \tag{5}$$

but from (4), $\mathrm{d}/\mathrm{d}\varrho(\mu H_z^2/8\pi) = \mu H_z^2/4\pi\varrho$, the square of the local Alfvén velocity, $A(\varrho)$, whereas $\mathrm{d}p/\mathrm{d}\varrho = a^2(\varrho)$, the square of the local velocity of sound. Then (5) reduces to the usual hydrodynamic equation for plane motion:

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\frac{a^2(\varrho)}{\varrho}\frac{\partial \varrho}{\partial x} + \delta\,\frac{\partial^2 v_x}{\partial x^2}\,, \tag{6}$$

with an effective local velocity of sound $a'(\varrho)$, given by $a'^2(\varrho) = a^2(\varrho) + A^2(\varrho)$. If we place a subscript on quantities referring to the undisturbed fluid in a state of equilibrium, then $H_z/\varrho = H_0/\varrho_0$, $A^2(\varrho) = (\mu H_0^2/4\pi\varrho_0)(\varrho/\varrho_0)$ and, for a perfect fluid, $a^2(\varrho) = \gamma(p_0/\varrho_0)(\varrho/\varrho_0)^{\gamma-1}$. We always have $\gamma < 2$, so $A(\varrho)$ increases more rapidly than $a(\varrho)$ with increasing compression ratio, ϱ/ϱ_0.

We now have to solve the system of two equations, (2) and (6), in two unknowns, v_x and ϱ. This is the same system as is treated by the theory of the propagation of finite hydrodynamic waves; so we may take over the results of this theory. For example, when the dissipation term is negligible, ($\delta \simeq 0$), Riemann's solution is valid and it predicts the formation of discontinuities in ϱ, which will be accompanied by discontinuities in H_z, as a consequence of equ. (4).

We may note that the equivalent hydrodynamic fluid is not a perfect fluid because its sound velocity $a'(\varrho)$ does not have the form $a'^2(\varrho) = \varkappa\varrho^{(\gamma'-1)}$, with $\varkappa$, γ' constants. However, if the initial magnetic field is sufficiently strong for a^2 to be negligible in comparison with A^2, then equations (6) and (2) reduce to the same form as for a perfect fluid with $\gamma = 2$ in the absence of a magnetic field and the solution will be mathematically the same.

The effect of finite electrical conductivity will be that the magnetic field

will tend to diffuse out of the disturbed region. The time, τ, required for this diffusion is of order of magnitude $\tau = L^2 4\pi\mu\sigma/c^2$, where L is of the order of magnitude of the linear dimension of the disturbed region. The effects of finite conductivity will be negligible as far as the formation of the shock is concerned if τ is greater than the time required for the build-up of a shock from the finite disturbance. The latter time is not greater than L/v, where v is the maximum fluid velocity in the disturbed region. We may consequently regard the conductivity as effectively infinite, if $v > \eta/L$, where $\eta = c^2/4\pi\mu\sigma$. [For mercury $\eta \simeq 8 \cdot 10^3$ cm^2/s]. For the interior of the sun, $\eta = 10^2$ cm^2/s, according to COWLING and the condition, in this case, is $v > 10^2/L$ cm^2/s, which can easily be satisfied even by relatively small linear dimensions.

These considerations show how magnetic shock-waves may arise, in a simple case in which there is a strong analogy with ordinary shock-waves. In the general case, in which the direction of the magnetic field is not normal to the direction of propagation, the analogy with finite hydrodynamic waves no longer holds and the solution is far more complex. The simple case treated here, however, may be useful as a first approach to the more general problem.

* * *

I am grateful to Dr. A. BAÑOS for many helpful discussions on this work.

RIASSUNTO

Si discute una soluzione delle equazioni (non lineari dipendenti dal tempo) della magnetoidrodinamica per la propagazione di onde piane di ampiezza finita, nel caso semplice in cui vi è una forte analogia con l'idrodinamica.

Applicazione del metodo del campo autoconsistente alle molecole di cis- e trans-dicloroetilene e di tetracloroetilene.

G. Favini e M. Simonetta

Istituto di Chimica Industriale, Laboratorio di Chimica Fisica dell'Università - Milano

(ricevuto l'11 Giugno 1958)

Riassunto. — È stato applicato il metodo SCF LCAO MO alle molecole di cis- e trans-dicloroetilene e di tetracloroetilene. Vengono calcolate e discusse alcune grandezze fisiche relative a queste molecole.

Nel corso di ricerche riguardanti l'applicazione del metodo degli orbitali molecolari a molecole coniugate contenenti uno o più eteroatomi, abbiamo applicato il metodo del campo autoconsistente (S.C.F.) descritto da Roothaan [1] a molecole clorurate semplici, adottando l'approssimazione π introdotta da Parr e Mulliken [2] in un lavoro sul butadiene, ammettendo cioè che tutti gli elettroni σ formino un cuore rigido non polarizzabile nel campo del quale si muovono gli elettroni appartenenti ad orbitali π.

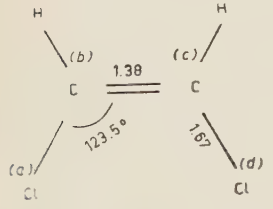

Fig. 1. – Cis-dicloro-etilene.

Dopo aver esaminato in un lavoro precedente a questo il cloruro di vinile [3], il metodo è stato da noi applicato alle seguenti molecole contenenti due o più atomi di cloro: cis-dicloroetilene; trans-dicloroetilene; tetracloroetilene.

La geometria delle molecole è stata desunta dai dati di diffrazione elettronica riportati in letteratura [4] ed è indicata nelle Fig. 1, 2, e 3.

---

[1] C. C. J. Roothaan: *Rev. Mod. Phys.*, **23**, 69 (1951).

[2] R. G. Parr e R. S. Mulliken: *Journ. Chem. Phys.*, **18**, 1338 (1950).

[3] M. Simonetta, G. Favini e S. Carrà: *Molecular Physics*, **1**, 181 (1958).

[4] L. O. Brockway, J. Y. Beach e L. Pauling: *Journ. Am. Chem. Soc.*, **57**, 2693 (1935); H. de Laszlo: *Nature*, **135**, 474 (1935).

Gli orbitali molecolari ψ_i sono stati ottenuti come combinazione lineare rispettivamente di quattro e di sei orbitali atomici, assunti come orbitali di Slater $2p_\pi$ per gli atomi di carbonio e $3p_\pi$ per gli atomi di cloro con $Z_C = 3.18$ e $Z_{Cl} = 6.099$.

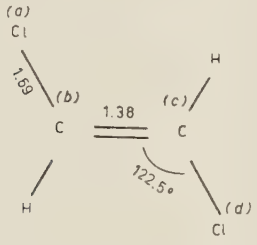

Fig. 2. – Trans-dicloro-etilene.

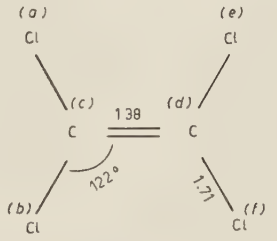

Fig. 3. – Tetracloro-etilene.

I valori numerici degli integrali su orbitali atomici impiegati nel calcolo sono riportati nelle Tabelle I e II.

Gli integrali di sovrapposizione sono stati ricavati per interpolazione dalle tabelle di Mulliken, Riecke, Orloff e Orloff [5].

Gli integrali di penetrazione a due centri sono stati calcolati con una estensione del metodo delle sfere tangenti uniformemente cariche [6], metodo precedentemente usato da altri autori per un calcolo approssimato di alcuni integrali di repulsione elettronica [7]; il loro valore è stato ottenuto dalla somma algebrica delle interazioni elettrostatiche dell'elettrone nell'orbitale π con il nucleo e con gli elettroni periferici dell'atomo, immaginando quelli degli strati più interni concentrati nel nucleo.

Gli integrali di penetrazione a tre centri e quelli relativi a due differenti orbitali sono stati ottenuti con la formula:

$$(a : bc) - \tfrac{1}{2} S_{bc}[(a : bb) + (a : cc)] .$$

Gli integrali di autopenetrazione sono invece stati calcolati per via teorica secondo il metodo di Goeppert-Mayer e Sklar [6,8].

Gli integrali di cuore I_{pq} sono stati calcolati dalla relazione:

$$I_{pq} = W_q S_{pq} - \sum_{r \neq q} [(r : pq) + n_r(rr/pq)] ,$$

[5] R. S. Mulliken, C. A. Riecke, D. Orloff e H. Orloff: *Journ. Chem. Phys.*, **17**. 1248 (1949).

[6] M. Simonetta, V. Pierpaoli e G. Favini: *Rend. Acc. Naz. Lincei*, **24**, 58 (1958).

[7] R. Pariser e R. G. Parr: *Journ. Chem. Phys.*, **21**, 466 (1953).

[8] M. Goeppert-Mayer e A. L. Sklar: *Journ. Chem. Phys.*, **6**, 645 (1938).

TABELLA I. – *Dicloroetileni: integrali su orbitali atomici* (*).

Integrali	Valori cis	Valori trans
Sovrapposizione:		
S_{ab}	0.172	0.166
S_{ac}	0.022	0.022
S_{bc}	0.266	
S_{ad}	0	
Penetrazione:		
$a : bb$	1.488	1.440
$b : aa$	0.280	0.230
$b : cc$	0.636	
$a : dd$	0	
$a : cc$	0.338	
$c : aa$	0.028	
$a : bc$	0.243	0.236
$a : bd$	0.016	
$a : cd$	0.029	0.028
$b : cd$	0.057	0.055
$b : ac$	0.010	
$b : ad$	0	
$a : ab$	2.579	2.486
$b : ab$	0.058	1.982
$a : ac$	0.317	
$b : bc$	3.231	
$a : ad$	0	
$b : bd$	0.260	
$a : aa$	28.506	
$b : bb$	23.654	
Integrali di cuore:		
$-I_{aa}$	47.677	45.606
$-I_{bb}$	47.917	47.729
$-I_{ab}$	11.478	10.905
$-I_{bc}$	16.875	16.825
$-I_{ad}$	0	
I_{ac}	1.515	1.493

Integrali	Valori cis	Valori trans
Coulombiani:		
aa/aa	15.480	
bb/bb	16.930	
aa/bb	7.500	7.430
aa/cc	5.006	
aa/dd	4.242	3.266
bb/cc	8.903	
Scambio:		
ab/ab	0.351	0.326
ab/ac	0.035	
ab/ad	0	
ab/bc	0.439	0.422
ab/bd	0.032	0.030
ab/cd	0.171	0.153
ac/ac	0.005	
ac/ad	0	
ac/bc	0.056	
ac/bd	0.003	
ad/ad	0	
ad/bc	0	
bc/bc	0.914	
Ibridi:		
aa/ab	1.976	1.902
aa/ac	0.225	
aa/ad	0	
aa/bc	1.663	1.654
aa/bd	0.129	0.118
aa/cd	0.795	0.687
bb/ab	2.101	2.022
bb/ac	0.180	
bb/ad	0	
bb/bc	3.436	
bb/bd	0.241	
bb/cd	1.196	1.154

(*) Gli integrali di sovrapposizione sono adimensionali: tutti gli altri sono in eV.

TABELLA II. – *Tetracloroetilene: integrali su orbitali atomici* (*).

Integrali	Valore	Integrali	Valore
Sovrapposizione:		Scambio:	
S_{ab}	0.012	ab/ab	0.001
S_{ac}	0.160	ac/ac	0.302
S_{ad}	0.021	ad/ad	0.005
S_{cd}	0.266	ab/ac	0.017
Penetrazione:		ab/ad	0.002
$a : bb$	0.172	ab/cd	0.020
$a : cc$	1.373	ab/ce	0.001
$c : aa$	0.241	ab/de	0.008
$a : dd$	0.260	ab/ef	0.001
$d : aa$	0.011	ac/ad	0.031
$c : dd$	0.636	ac/bc	0.232
$a : aa$	28.506	ac/cd	0.406
$c : cc$	23.654	ac/ce	0.028
$a : bc$	0.124	ac/cf	0.027
$a : bd$	0.005	ac/bd	0.022
$a : cd$	0.217	ac/de	0.148
$a : ce = a : cf$	0.014	ac/df	0.141
$a : de = a : df$	0.021	ad/bd	0.003
$c : ab$	0.003	ad/cd	0.053
$c : ad$	0.009	ad/ce	0.003
$c : de = c : df$	0.052	ad/cf	0.003
$a : ab$	0.172	cd/cd	0.914
$a : ac$	2.390	Ibridi:	
$a : ad$	0.302	aa/ab	0.121
$c : ac$	1.912	aa/ac	1.827
$c : dc$	3.231	aa/ad	0.215
$c : ec$	0.248	aa/bc	0.962
Integrali di cuore:		aa/bd	0.101
I_{aa}	— 63.489	aa/cd	1.640
I_{cc}	— 73.661	aa/ce	0.122
I_{ab}	— 1.061	aa/cf	0.111
I_{ac}	— 14.029	aa/de	0.740
I_{ad}	— 1.884	aa/df	0.657
I_{cd}	— 23.725	aa/ef	0.045
Coulombiani:		cc/ab	0.088
aa/aa	15.480	cc/ac	1.943
cc/cc	16.930	cc/ad	0.171
aa/bb	4.664	cc/cd	3.436
aa/cc	7.356	cc/ce	0.230
aa/dd	4.973	cc/de	1.110
aa/ee	4.275	cc/ef	0.060
aa/ff	3.239		
cc/dd	8.903		

(*) Gli integrali non riportati in tabella sono nulli.

in cui W_a è il potenziale di prima ionizzazione per l'elettrone $2p$ del carbonio o il potenziale di seconda ionizzazione per l'elettrone $3p$ del cloro, assunti rispettivamente uguali a -11.54 eV e -26.38 eV ([3]), ed n_r è la carica dell'atomo r del cuore.

Gli integrali coulombiani sono stati valutati con il metodo delle sfere uniformemente cariche ad eccezione di quelli monocentrici ottenuti per via teorica ([9,6]).

Per gli integrali di scambio e per quelli ibridi si è adottato lo stesso criterio visto per quelli di penetrazione, cioè si è usata la formula:

$$(cc/ab) = \tfrac{1}{2} S_{ab}[(cc/aa) + (cc/bb)] .$$

Gli orbitali di partenza sono stati calcolati col metodo LCAO standard assumendo per gli integrali coulombiani e di scambio i valori impiegati da GOLDSTEIN ([10]) per il cloruro di vinile, e precisamente:

$$H_{Cl} = -10.64 \text{ eV}, \quad H_C = -8.14 \text{ eV}, \quad \beta_{C\,Cl} = -2.45 \text{ eV}, \quad \beta_{C\,C} = -5 \text{ eV},$$

mentre per gli integrali di sovrapposizione sono stati adoperati i valori riportati nelle Tabelle I e II.

Al posto degli orbitali atomici φ_p sono stati assunti per i dicloroetileni gli orbitali di simmetria normalizzati ([11]):

$$\sigma_1 = \frac{1}{N_1}(\varphi_a + \varphi_d), \qquad \text{con } N_1 = 1.4142,$$

$$\sigma_2 = \frac{1}{N_2}(\varphi_b + \varphi_c), \qquad \text{con } N_2 = 1.5912,$$

$$\sigma_3 = \frac{1}{N_3}(\varphi_b - \varphi_c), \qquad \text{con } N_3 = 1.2116,$$

$$\sigma_4 = \frac{1}{N_4}(\varphi_a - \varphi_d), \qquad \text{con } N_4 = 1.4142,$$

([9]) R. G. PARR e B. L. CRAWFORD Jr.: *Journ. Chem. Phys.*, **16**, 1049 (1948).

([10]) J. H. GOLDSTEIN: *Journ. Chem. Phys.*, **24**, 507 (1956).

([11]) I fattori di normalizzazione sono stati calcolati dagli integrali di sovrapposizione S_{pq} delle Tabelle I e II. Per es. per il tetracloroetilene:

$$N_1 = [4 + 2S_{ab} + 2S_{ae} + 2S_{af} + 2S_{be} + 2S_{bf} + S_{ef}]^{\frac{1}{2}} = (4.048)^{\frac{1}{2}} = 2.012 .$$

e per il tetracloroetilene:

$$\sigma_1 = \frac{1}{N_1}(\varphi_a + \varphi_b + \varphi_e + \varphi_f), \qquad \text{con } N_1 = 2.0120,$$

$$\sigma_2 = \frac{1}{N_2}(\varphi_c + \varphi_d), \qquad \text{con } N_2 = 1.5912,$$

$$\sigma_3 = \frac{1}{N_3}(\varphi_a + \varphi_b - \varphi_e - \varphi_f), \qquad \text{con } N_3 = 2.0120,$$

$$\sigma_4 = \frac{1}{N_4}(\varphi_c - \varphi_d), \qquad \text{con } N_4 = 1.2116,$$

$$\sigma_5 = \frac{1}{N_5}(\varphi_a - \varphi_b + \varphi_e - \varphi_f), \qquad \text{con } N_5 = 1.9880,$$

$$\sigma_6 = \frac{1}{N_6}(\varphi_a - \varphi_b - \varphi_e + \varphi_f). \qquad \text{con } N_6 = 1.9880.$$

Le Tabelle III e IV riportano gli integrali sugli orbitali di simmetria σ rispettivamente per i dicloroetileni e per il tetracloroetilene. Gli orbitali molecolari risulteranno allora del tipo:

$$\psi_i = \sum_n d_{in}\,\sigma_n$$

TABELLA III. – *Dicloroetileni: integrali su orbitali di simmetria.*

Integrali	Valore cis	Valore trans	Integrali	Valore cis	Valore trans
Sovrapposizione:			Scambio e ibridi:		
S_{12}	0.172	0.167	12/11 = 12/44	1.389	1.304
S_{34}	0.175	0.168	12/22 = 12/33	1.652	1.598
			34/11 = 34/44	1.411	1.312
Integrali di cuore:			34/22 = 34/33	1.678	1.608
			12/12	0.262	0.244
			34/34	0.270	0.243
$I_{11} = I_{44}$	— 47.677	— 45.606	12/34	0.267	0.244
I_{22}	— 51.178	— 50.991	14/24	0.567	0.587
I_{33}	— 42.290	— 42.104	13/23	0.092	0.089
I_{12}	— 11.549	— 11.019	14/23	1.294	1.257
I_{34}	— 11.626	— 10.986	13/13	0.120	0.112
			14/14	5.619	6.107
Coulombiani:			23/23	4.319	
			24/24	0.074	0.073
11/11 = 44/44	9.861	9.373	13/14	0.633	0.647
22/22 = 33/33	12.916		23/24	0.389	0.372
11/22	6.252	6.218	13/23	0.585	0.562

TABELLA IV. – *Tetracloroetilene: integrali su orbitali di simmetria.*

Integrali	Valore	Integrali	Valore
Sovrapposizione:		Scambio e ibridi:	
		12/12	0.411
S_{12}	0.226	12/34	0.414
S_{34}	0.228	34/34	0.418
		13/13	3.505
Integrali di cuore:		13/24	1.235
		24/24	4.315
$I_{11} = I_{33}$	— 63.785	14/14	0.169
I_{22}	— 76.924	23/23	0.095
I_{44}	— 68.033	14/23	0.125
$I_{66} = I_{55}$	— 63.186	14/24	0.760
I_{12}	— 19.882	14/13	0.537
I_{34}	— 19.929	13/23	0.461
		23/24	0.501
Coulombiani:		15/25 = 36/46	0.340
		35/45 = 16/26	0.280
11/11 = 33/33	6.914	15/15 = 36/36	2.963
55/55 = 66/66		16/16 = 35/35	2.445
22/22 = 44/44	12.916	25/25 = 46/46	0.040
11/22	6.165	45/45 = 26/26	0.033
		56/56	3.522

e il calcolo iterativo è stato condotto fino ad autoconsistenza dei valori dei d_{in} a ± 0.0001.

La Tabella V riporta i valori dei d_{in} e delle energie (ε_i) degli orbitali molecolari autoconsistenti per lo stato fondamentale. Dai valori della tabella si ottengono in ordine di energia crescente i seguenti quattro orbitali molecolari ortonormali per i dicloroetileni:

$$\psi_1 = \begin{Bmatrix} 0.41765 \\ 0.41027 \end{Bmatrix} (\varphi_a + \varphi_d) + \begin{Bmatrix} 0.44715 \\ 0.45457 \end{Bmatrix} (\varphi_b + \varphi_c) ,$$

(valore superiore cis-),

$$\psi_2 = \begin{Bmatrix} 0.19103 \\ 0.18703 \end{Bmatrix} (\varphi_b - \varphi_c) + \begin{Bmatrix} 0.65984 \\ 0.66232 \end{Bmatrix} (\varphi_a - \varphi_d) ,$$

$$\psi_3 = \begin{Bmatrix} 0.58390 \\ 0.58824 \end{Bmatrix} (\varphi_a + \varphi_d) - \begin{Bmatrix} 0.45510 \\ 0.44683 \end{Bmatrix} (\varphi_b + \varphi_c) ,$$

(valore inferiore trans-),

$$\psi_4 = \begin{Bmatrix} 0.81623 \\ 0.81609 \end{Bmatrix} (\varphi_b - \varphi_c) - \begin{Bmatrix} 0.28352 \\ 0.27543 \end{Bmatrix} (\varphi_a - \varphi_d) ,$$

TABELLA V. – *Orbitali molecolari autoconsistenti per lo stato fondamentale.*

Energie orbitali (eV)	Dicloroetilene cis	Dicloroetilene trans	Energie orbitali (eV)	Tetracloroetilene
ε_1	— 14.95088	— 14.79800	ε_1	— 16.38623
ε_2	— 12.24408	— 12.10709	ε_2	— 13.23239
ε_3	— 7.92186	— 7.93950	ε_3	— 11.29902
ε_4	+ 6.69358	+ 6.58584	ε_4	— 11.21991
			ε_5	— 7.08470
			ε_6	+ 6.89474
Coefficienti d_{in}:			Coefficienti d_{in}:	
d_{11}	0.59064	0.58020	d_{11}	0.55602
d_{12}	0.71150	0.72331	d_{12}	0.71496
d_{23}	0.23145	0.22661	d_{23}	0.89544
d_{24}	0.93315	0.93666	d_{24}	0.28559
d_{31}	0.82575	0.83190	$d_{35} = d_{46}$	1.—
d_{32}	— 0.72415	— 0.71099	d_{51}	0.86294
d_{43}	0.98895	0.98878	d_{52}	— 0.73666
d_{44}	— 0.40096	— 0.38951	d_{63}	0.50300
			d_{64}	— 0.98654

e i seguenti sei orbitali per il tetracloroetilene:

$$\psi_1 = 0.27635(\varphi_a + \varphi_b + \varphi_e + \varphi_f) + 0.44932(\varphi_c + \varphi_d)\,,$$
$$\psi_2 = 0.44505(\varphi_a + \varphi_b - \varphi_e - \varphi_f) + 0.23571(\varphi_c - \varphi_d)\,,$$
$$\psi_3 = 0.50302(\varphi_a - \varphi_b + \varphi_e - \varphi_f)\,,$$
$$\psi_4 = 0.50302(\varphi_a - \varphi_b - \varphi_e + \varphi_f)\,,$$
$$\psi_5 = 0.42890(\varphi_a + \varphi_b + \varphi_e + \varphi_f) - 0.46296(\varphi_c + \varphi_d)\,,$$
$$\psi_6 = 0.25000(\varphi_a + \varphi_b - \varphi_e - \varphi_f) - 0.81425(\varphi_c - \varphi_d)\,.$$

Sono state inoltre calcolate le differenze di energia fra lo stato eccitato e quello fondamentale, corrispondenti alla prima transizione π-π, mediante le relazioni:

$$E_{T_{ij}} - E_N = \varepsilon_j - \varepsilon_i - J_{ij}\,,$$
$$E_{V_{ij}} - E_N = \varepsilon_j - \varepsilon_i - J_{ij} + 2K_{ij} = E_{T_{ij}} - E_N + 2K_{ij}\,;$$

le densità degli elettroni π dall'espressione $q_p = 2\sum_i c_{ip}^2$ e gli ordini di legame dalla formula $n_{pq} = 2\sum_i c_{ip}\,c_{iq}\,S_{pq}$ (l'indice i è relativo ai soli orbitali moleco-

lari occupati); i potenziali teorici di prima ionizzazione (I) che secondo MULLIKEN [12] corrispondono praticamente, a meno del segno, all'energia dell'ultimo orbitale occupato; le distanze interatomiche C—C applicando la formula di Coulson [13] e quelle C—Cl applicando la formula di Gordy [14,3]: i dati ottenuti sono raggruppati nelle Tabelle VI e VII e confrontati con i dati sperimentali a disposizione.

Per la molecola del cis-dicloroetilene, mediante i valori delle densità degli elettroni π e degli ordini di legame è stato calcolato il momento dipolare di risonanza che è stato poi composto con il momento dovuto ai legami σ, per i quali abbiamo adottato i valori: $\bar{C}—\overset{+}{H} = 0.4$ Debye e $\overset{+}{C}—\bar{Cl} = 1.5$ Debye [15].

L'accordo tra dati teorici e sperimentali è da ritenersi senz'altro soddisfacente.

Il calcolo del momento elettrico dipolare è stato eseguito con gli stessi criteri anche per il cloruro di vinile ottenendo un valore di 1.63 D (sperimentale 1.44 D [16]).

TABELLA VI. – *Dicloroetileni.*

	cis-		trans-	
	teor.	sperim.	teor.	sperim.
J_{34} (eV)	10.63443	—	10.40293	—
K_{34} (eV)	2.29040	—	2.25493	—
$E_{V_{34}} - E_N$ (eV)	8.56181	—	8.63227	—
$E_{T_{34}} - E_N$ (eV)	3.98101	—	4.12241	—
$E_{TV_{34}} - E_N$ (eV)	6.27141	6.524 [18]	6.37734	6.356 [18]
q_{Cl}	1.90152	—	1.90603	—
q_C	0.88710	—	0.88254	—
n_{ab}	0.03238	—	0.03156	—
n_{ac}	— 0.01804	—	— 0.01762	—
n_{ad}	0	—	0	—
n_{bc}	0.39428	—	0.39507	—
I (eV)	7.92186	9.7 [18]	7.93950	9.9 [18]
R_{C-C} (A°)	1.451	1.38 [4]	1.451	1.38 [4]
R_{C-Cl} (A°)	1.753	1.67 [4]	1.753	1.69 [4]
μ (D)	2.09	1.89 [17]	0	0

[12] R. S. MULLIKEN: *Journ. Chem. Phys.*, **46**, 497 (1949).
[13] C. A. COULSON: *Proc. Roy. Soc.*, A **169**, 413 (1939).
[14] W. GORDY: *Journ. Chem. Phys.*, **15**, 305 (1947).
[15] J. W. SMITH: *Electric dipole moments* (London, 1955).
[16] C. P. SMYTH: *Dielectric Behavior and Structure* (New York, 1955).
[17] A. MARYOTT, M. E. HOBBS e P. M. GROSS: *Journ. Am. Chem. Soc.*, **63**, 659 (1941).
[18] A. D. WALSH: *Trans. Farad. Soc.*, **41**, 35 (1945).

TABELLA VII. – *Tetracloroetilene.*

		teor.	sperim.
J_{56}	(eV)	10.85982	—
K_{56}	(eV)	2.58680	—
$E_{V_{56}} - E_N$	(eV)	8.29322	—
$E_{T_{56}} - E_N$	(eV)	3.11962	—
$\overline{E}_{TV_{56}} - E_N$	(eV)	5.70642	6.292 [18]
q_{C1}		1.92890	—
q_C		0.94356	—
n_{ab}		−0.00229	—
n_{ac}		0.01953	—
n_{ad}		−0.01506	—
n_{ae}		0	—
n_{cd}		0.38374	—
I	(eV)	7.0847	9.5 [18]
R_{C-C}	(A°)	1.453	1.38 [4]
R_{C-Cl}	(A°)	1.756	1.71 [4]
μ	(D)	0	0

SUMMARY

A SCF LCAO MO calculation for the cis- and trans-dichloroethylene and tetrachloroethylene is presented. Ionization potential, π-π transition energy, dipole moment and bond lenghts values are calculated and discussed.

Uncertainty Relation and Diffraction by a Slit.

G. Beck and H. M. Nussenzveig

Centro Brasileiro de Pesquisas Físicas - Rio de Janeiro, Brasil

(ricevuto il 13 Giugno 1958)

Summary. — Arguments based on the rigorous theory of diffraction of a plane wave by a slit of arbitrary width are applied to discuss the uncertainties as a function of the distance from the slit.

1. – The diffraction of a plane wave by a slit has often been discussed as an illustration of Heisenberg's uncertainty relations and their role in the process of measurement. Such a discussion involves two distinct steps: *A*) the evaluation of the uncertainties, based on the solution of a well defined boundary value problem, representing schematically the example in question; *B*) the analysis of the connection between the uncertainties and the process of measurement. Step *B* is related with many apparently still unsettled questions. In the case of the above mentioned example, however, even the treatment of step *A* has not been satisfactory. It has been restricted to a wide slit [1], and only elementary diffraction theory has been employed; application of the results to a narrow slit has led to misunderstandings (see Sect. 5). The purpose of this paper is to give an improved treatment of step *A*, in the case of a slit of arbitrary width. We shall not attempt to enter into the discussion of step *B*.

2. – Let us consider a monochromatic plane wave of angular frequency ω, perpendicularly incident on an infinite slit of width $2a$ in a perfectly reflecting

[1] N. Bohr: *Phys. Rev.*, **48**, 696 (1935).

screen of vanishing thickness (the effect of non-zero thickness will be discussed later). We shall employ the co-ordinate system shown in Fig. 1. The time factor $\exp[-i\omega t]$ will be omitted throughout. The incident wave, $u_0 = A\exp[ikz]$, may represent either the Schrödinger wave function of a particle of mass m, in which case $\omega = \hbar k^2/2m$, or else it may represent the electric field amplitude of an electromagnetic wave (linearly polarized parallel to the edges), in which case $\omega = ck$. The rigorous formulation of the problem is the same in both cases. The total wave function $u(y, z)$ is given by

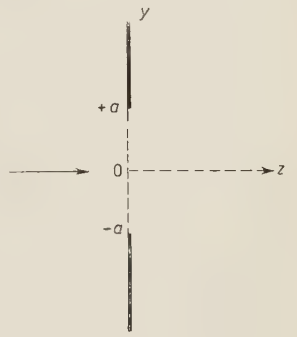

Fig. 1. – The co-ordinate system.

$$u(y,z) = \begin{cases} A(\exp[ikz] - \exp[-ikz]) + \varphi(y,-z) & (z \leqslant 0), \\ \varphi(y,z) & (z \geqslant 0), \end{cases} \tag{1}$$

where $\varphi(y, z)$, defined for $z \geqslant 0$, satisfies the following conditions (²):

(i) $(\Delta + k^2)\varphi = 0$; (ii) $\varphi(y, 0) = 0$ $(|y| \geqslant a)$; (iii) $(\partial\varphi/\partial z)(y, 0) = ikA$ $(|y| < a)$; (iv) φ satisfies Sommerfeld's radiation condition at infinity; (v) φ is everywhere finite; (vi) $\nabla\varphi$ is quadratically integrable over any domain of three-dimensional space, including the edges of the slit.

If we express $\varphi(y, 0)$ as a Fourier integral,

$$\varphi(y, 0) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \chi(k_y, 0) \exp[ik_y y]\,\mathrm{d}k_y\,, \tag{2}$$

we find, taking into account condition (ii),

$$\chi(k_y, 0) = (2\pi)^{-\frac{1}{2}} \int_{-a}^{+a} \varphi(y, 0) \exp[-ik_y y]\,\mathrm{d}y\,, \tag{3}$$

and, employing conditions (i) and (iv), we get (*)

$$\varphi(y, z) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \chi(k_y, z) \exp[i\,k_y y]\,\mathrm{d}k_y\,, \tag{4}$$

(²) C. J. Bouwkamp: *Rep. Progr. Phys.*, **17**, 35 (1954), p. 38.

(*) Eq. (4) is equivalent to Eq. (2.23) of reference (²).

where

$$\chi(k_y, z) = \chi(k_y, 0) \exp [ik_z z] , \tag{5}$$

$$k_z = (k^2 - k_y^2)^{\frac{1}{2}} ; \quad \mathrm{Im}\,(k_z) \leqslant 0 . \tag{6}$$

The second member of (4) represents a superposition of plane waves, travelling in all directions (for $k_y < k$), and evanescent waves, exponentially attenuated in the z direction (for $k_y > k$).

In the case of non-relativistic particles, $|\varphi(y, \zeta)|^2$ and $|\chi(k_y, \zeta)|^2$ (with suitable normalization factors) may be interpreted as probability distributions in y and in k_y, respectively, on a given plane $z = \zeta$. The physical interpretation is more involved in the electromagnetic case.

According to (5) and (6), we have

$$|\chi(k_y, \zeta)|^2 = \begin{cases} |\chi(k_y, 0)|^2 & (\text{for } k_y < k), \\ |\chi(k_y, 0)|^2 \exp [-2(k_y^2 - k^2)^{\frac{1}{2}} \zeta] & (\text{for } k_y > k). \end{cases} \tag{7}$$

Therefore, the distribution function in k_y for travelling waves does not depend on ζ, whereas it decreases exponentially with ζ for evanescent waves.

3. – According to (3), (4) and (5), it suffices to know $\varphi(y, 0)$ on the slit in order to determine the solution. It may be shown that [3]

$$\varphi(y, 0) = \sum_{n=1}^{\infty} C_n (1 - y^2/a^2)^{n-\frac{1}{2}} . \tag{8}$$

In the case of a *very narrow slit* ($ka \ll 1$), the coefficients C_n decrease rapidly with n, and, for an incident wave of unit amplitude, $A = 1$, we may take (*): $C_1 = -ika$; $C_2 = C_3 = \ldots = 0$, so that

$$\varphi(y, 0) = -ika(1 - y^2/a^2)^{\frac{1}{2}} . \tag{9}$$

It follows from (8) that $\nabla\varphi$ has a singularity at the edges, where it becomes infinitely large as $D^{-\frac{1}{2}}$ (D denotes the distance from the edge). The same type of singularity appears in Sommerfeld's well known theory of diffraction by a half-plane.

---

(3) A. Sommerfeld: *Optics* (New York, 1954), p. 278.

(*) See reference (2), p. 74.

Replacing (9) in (3), we find

$$\chi(k_y, 0) = -i(\pi/2)^{\frac{1}{2}} k a^2 J_1(k_y a)/(k_y a) , \tag{10}$$

where $J_1(x)$ is Bessel's function of the first order. For $k_y a \gg 1$,

$$\chi(k_y, 0) \approx -ika^2 (k_y a)^{-\frac{3}{2}} \sin(k_y a - \pi/4) . \tag{11}$$

The asymptotic behaviour of $\chi(k_y, 0)$, given by (11), is entirely determined by the singularity at the edges. This follows from a general theorem on the asymptotic behaviour of Fourier integrals whose integrands have singularities at both ends of the interval of integration (4).

The curves of $|\varphi(y, 0)|^2$ and $|\chi(k_y, 0)|^2$, according to (9) and (10), are shown in Fig. 2*a* and 2*b*.

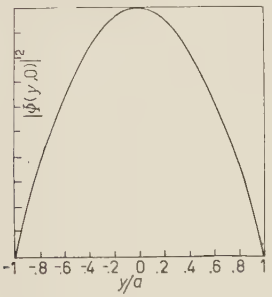

Fig. 2*a*. – Distribution function in y for $\zeta = 0$.

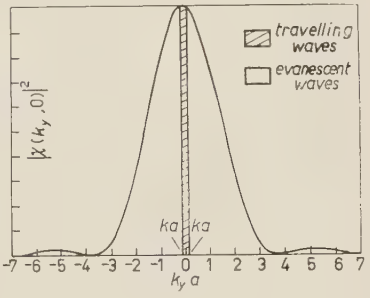

Fig. 2*b*. – Distribution function in k_y for $\zeta = 0$.

The wave function at large distances ϱ from the origin is given by (*)

$$\varphi \approx \tfrac{1}{4}(ka)^2 \cos\theta (2\pi/k\varrho)^{\frac{1}{2}} \exp[i(k\varrho - 3\pi/4)] , \tag{12}$$

where θ is the polar angle with respect to the z-axis. According to (12),

$$|\varphi(y, \zeta)|^2 = (\pi/8k)(ka)^4 \zeta^2 (y^2 + \zeta^2)^{-\frac{3}{2}} \qquad \text{for } k\zeta \gg 1. \tag{13}$$

The corresponding value of $|\chi(k_y, \zeta)|^2$ follows from (7) and (10). The results

---

(4) A. Erdélyi: *Asymptotic Expansions* (New York, 1956), p. 49. See also H. M. Nussenzveig: *Thesis* (S. Paulo, 1957, to be published).

(*) See reference (2), p. 74.

are shown in Fig. 3*a* and 3*b*. Fig. 4 shows the limiting form of $|\chi(k_y, \zeta)|^2$ for $\zeta \to \infty$.

Now let us consider the case of a *wide slit* ($ka \gg 1$). No simple rigorous expression for the wave function is known in this case. However, we can make

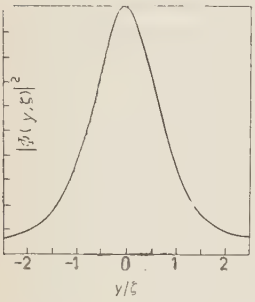

Fig. 3*a*. – Distribution function in y for $k\zeta \gg 1$.

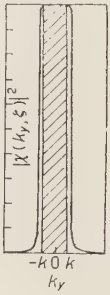

Fig. 3*b*. – Distribution function in k_y for $k\zeta \gg 1$.

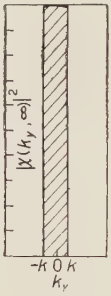

Fig. 4. – Distribution function in k_y for $\zeta \to \infty$.

the following assertions: *A*) $|\chi(k_y, 0)|^2$ has a very large peak, located at $k_y = 0$; the width of this peak is of the order of $1/a$. This may be attributed to the proximity to « geometrical optics » conditions (*). *B*) $|\chi(k_y, 0)|^2$ decreases asymptotically like k_y^{-3}. This is shown by the considerations which follow (11).

4. – The preceding results will now be applied to the evaluation of the « uncertainties » in y and in k_y. In the quantitative derivation of the uncertainty relation (5), the uncertainties are defined as root mean square deviations from the mean values. While this definition may be convenient in some cases, there exist other cases in which it does not lead to physically meaningful results; as will be seen below, this happens in the present problem. It may be more suitable, then, to apply a different definition (+). The uncertainty in a given variable may be defined, for instance, as the « half-width »

(*) See H. M. NUSSENZVEIG (4). In the immediate neighbourhood of $k_y = 0$, the main term of $|\chi(k_y, 0)|^2$ is given by Kirchhoff's approximation: $|\chi(k_y, 0)|^2 \simeq (2/\pi)a^2 \cdot \sin^2(k_y a)/(k_y a)^2$. Within the half-width of the peak, corrections to this term (J. B. KELLER: *Journ. Appl. Phys.*, **28**, 426 (1957)) may be neglected, for $ka \gg 1$. Even for $ka = 3$, comparison with the exact solution (S. N. KARP and A. RUSSEK: *Diffraction by a wide slit*, New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Research Report No. EM-75 (1955)) shows that Kirchhoff's approximation gives the half-width with an error of less than 30%.

(5) E. H. KENNARD: *Zeits. Phys.*, **44**, 326 (1927).

(+) This was pointed out to us by Prof. M. SCHÖNBERG.

of the probability distribution in that variable (assuming that the shape of the distribution allows a half-width to be defined). This definition has been applied in the theory of the natural line breadth (6). We shall see that it should also be preferred in the present problem.

Let us denote by $\Delta y(\zeta)$, $\Delta k_y(\zeta)$, the root mean square deviation from the mean values of y and k_y, respectively, evaluated on the plane $z=\zeta$ (*):

$$[\Delta y\,(\zeta)]^2 = \int_{-a}^{+a} y^2\,|\varphi(y\ ,\zeta)|^2\,\mathrm{d}y \Big/ \int_{-a}^{+a} |\varphi(y,\zeta)|^2\,\mathrm{d}y\ , \tag{14}$$

$$[\Delta k_y(\zeta)]^2 = \int_{-\infty}^{+\infty} k_y^2\,|\chi(k_y,\zeta)|^2\,\mathrm{d}k_y \Big/ \int_{-\infty}^{+\infty} |\chi(k_y,\zeta)|^2\,\mathrm{d}k_y\ . \tag{15}$$

In the case of a *narrow slit*, it follows from (9), (10) and (11) that

$$\Delta y(0) = a/\sqrt{5}\ , \tag{16}$$

$$\Delta k_y(0) \to \infty \ \text{logarithmically}\ , \tag{17}$$

$$\Delta y(0)\,\Delta k_y(0) \to \infty \ \text{logarithmically}\ . \tag{18}$$

The logarithmic divergence of $\Delta k_y(0)$ is obviously due to the asymptotic behaviour (11) of $\chi(k_y, 0)$, which is determined by the singularity at the edges. For a screen of finite thickness having « rounded » edges, the singularity would disappear and Δk_y would no longer diverge (7).

On a plane $z=\zeta$ such that $k\zeta \gg 1$, we find, according to (13), (7) and (10),

$$\Delta y(\zeta) \to \infty \ \text{logarithmically}\ , \tag{19}$$

$$\Delta k_y(\zeta) \simeq k/\sqrt{3}\ , \tag{20}$$

$$\Delta y(\zeta)\,\Delta k_y(\zeta) \to \infty \ \text{logarithmically}\ . \tag{21}$$

---

(6) See *e.g.* W. Heitler: *Quantum Theory of Radiation* (Oxford, 1954), 3rd ed., p. 184.

(*) These quantities have to be distinguished from the corresponding values averaged over the whole space, which are not appropriate to this problem.

(7) R. Gans and G. Beck: *Rev. Unión Mat. Argent.*, **14**, 425 (1950). Even for a screen of finite thickness having right angle corners at the edges Δk_y would no longer diverge. In fact, at such a corner, $\nabla\varphi$ behaves like $D^{-\frac{1}{3}}$ (D = distance from the corner), and the probability distribution in k_y behaves asymptotically like $k_y^{-10/3}$ (see H. M. Nussenzveig (4)).

The logarithmic divergence of $\Delta y(\zeta)$ also arises from the asymptotic behaviour of the probability distribution.

Both divergences, (17) and (19), reveal the inadequacy of the adopted definition of uncertainty. In fact, in both cases, the divergence arises, not from a lack of concentration of the probability distribution, but from the exaggerated weight which is attributed to large values of the variable, in spite of their extremely small probability.

On the other hand, inspection of Fig. 2-4 shows that the half-width will be a good measure of dispersion in the present case. Let us denote by $\delta y(\zeta)$, $\delta k_y(\zeta)$, the half-widths of the probability distributions in y and in k_y, respectively, on the plane $z=\zeta$. Then, according to (9) and (10),

$$\delta y(0) = \sqrt{2}\, a \,, \tag{22}$$

$$\delta k_y(0) \simeq 3.2/a \,, \tag{23}$$

$$\delta y(0)\, \delta k_y(0) \simeq 4.6 \,, \tag{24}$$

which satisfies the uncertainty relation. The small value of the uncertainty product (24), in contrast with (18), agrees with our expectation.

It follows from (13), (7) and (10) that, for $k\zeta \gg 1$,

$$\delta y(\zeta) \simeq 1.5\zeta \,, \tag{25}$$

$$\delta k_y(\zeta) \simeq 2k \,, \tag{26}$$

$$\delta y(\zeta)\, \delta k_y(\zeta) \simeq 3.1 k\zeta \gg 1 \,. \tag{27}$$

The increase of $\delta y(\zeta)$ with ζ (25) reflects the linear spread of the beam in the region $k\zeta \gg 1$.

Let us consider now the case of a *wide slit*. Two properties of $|\chi(k_y, 0)|^2$ in this case were given in Sect. 3. According to property B, (17) still holds for a wide slit (in a screen of vanishing thickness). On the other hand, according to property A,

$$\delta k_y(0) \sim 1/a \,, \tag{28}$$

whereas we obviously have

$$\delta y(0) \sim a \,, \tag{29}$$

so that

$$\delta y(0)\, \delta k_y(0) \sim 1 \,. \tag{30}$$

Since the large peak of $|\chi(k_y, 0)|^2$ belongs to the spectrum of travelling waves, it follows from (7) that, in contrast with the case of a narrow slit, δk_y does not depend on ζ:

$$\delta k_y(\zeta) = \delta k_y(0) \sim 1/a \qquad \text{(for any } \zeta\text{)}. \tag{31}$$

5. – In the usual analysis of the connection between diffraction by a slit and the uncertainty relation (8), the uncertainty in k_y is evaluated as the width of the main peak in the Fraunhofer diffraction pattern (according to elementary diffraction theory). For a *wide slit*, the result corresponds, in our notation, to

$$\delta k_y(\infty) \sim 1/a\,. \tag{32}$$

It follows from (31) that, in (30), $\delta k_y(0)$ may be replaced by $\delta k_y(\infty)$ *in the case of a wide slit.*

From the physical point of view, (31) means that the wave function beyond a wide slit behaves approximately as a superposition of isochromatic plane waves in free space (for which the distribution function in k_y would be rigorously independent of ζ). This is no longer true for a narrow slit, because in this case border effects are predominant over the entire region of the slit, and the analogy with waves in free space breaks down.

On the other hand, for a slit of arbitrary width, we have

$$\delta k_y(\infty) \leqslant 2k\,, \tag{33}$$

so that

$$\delta y(0)\,\delta k_y(\infty) \lesssim 2ka\,. \tag{34}$$

It follows that, for a *very narrow slit*,

$$\delta y(0)\,\delta k_y(\infty) \ll 1\,. \tag{35}$$

If $\delta k_y(\infty)$ could be replaced by $\delta k_y(0)$, as in (32), (35) would contradict the uncertainty relation (9). However, for a very narrow slit, this replacement

(8) W. HEISENBERG: *The Physical Principles of the Quantum Theory* (Chicago, 1930), p. 23.

(9) D. I. BLOHINČEV: *Grundlagen der Quantenmechanik* (Berlin, 1953), p. 50. According to BLOHINČEV, the solution of this paradox lies in the fact that one cannot ascribe a definite wavelength to the field beyond a very narrow slit. Plane waves of

is clearly not permissible (compare (23) and (26), and Fig. 2*b* and 4). Therefore, (35), which relates the uncertainties in two different planes, does not contradict the uncertainty relation.

different wavelengths would indeed appear in the total (three-dimensional) Fourier representation of the wave function. In the present problem, however, only the Fourier representation in y can be employed. In this representation, $\lambda = 2\pi/k(\omega)$ is a well defined constant, and $k_y = k \sin\theta$ (where θ is no longer restricted to real values only).

RIASSUNTO (*)

Argomenti basati sulla teoria rigorosa della diffrazione di un'onda piana su una fenditura di larghezza arbitraria si applicano alla discussione delle indeterminazioni come funzioni della distanza dalla fenditura.

(*) *Traduzione a cura della Redazione.*

Angular Distribution in π-μ Decay (*).

P. Connolly and G. Lynch

Laboratory of Nuclear Studies, Cornell University - Ithaca, New York

(ricevuto il 16 Giugno 1958)

Summary. — Analysis has been made of the decays of 5034 photoproduced π^+ mesons stopping in nuclear emulsion. The angular distribution consistent with isotropy.

As a result of a suggestion by Lattes (1) last year that there may be not isotropy in π-μ decay, renewed interest developed in this subject. A group of Rumanian investigators (2) reported a large forward-backward asymmetry in π-μ decay. Bruin (3) in Holland also reported evidence pointing toward anisotropy. In contrast to these emulsion experiments, two counter experiments (4,5) have found isotropy. Furthermore Cassels (6) has reported that other workers have come to the conclusion that their emulsion data are consistent with isotropy. A recent Russian experiment (7) with good statistics

(*) Supported in part by the joint program of the Office of Naval Research and the U.S. Atomic Energy Commission.

(1) C. M. G. Lattes: *Report at Varenna Conference on Cosmic Rays*, June 1957 (Private communication).

(2) H. Hulubei, J. Auslander, E. Balea, E. Friedlander, S. Titeica and T. Visky: *Compt. Rend. Acad. Sci. (Paris)*, **245**, 1037 (1957).

(3) F. Bruin and M. Bruin: *Physica*, **23**, 551 (1957).

(4) R. L. Garwin, G. Gidal, L. M. Lederman and M. Weinrich: *Phys. Rev.*, **108**, 1589 (1957).

(5) A. V. Crewe, V. E. Kruse, R. H. Miller and L. G. Pondrom: *Phys. Rev.*, **108**, 1531 (1957).

(6) J. M. Cassels: *Nature*, **180**, 1245 (1958).

(7) N. P. Bogačev, A. C. Mihul, M. G. Petrasco and V. M. Sidorov: *Žu. Éksper. Teor. Fiz.*, **34**, 531 (1958).

showed no forward-backward asymmetry. These experiments are worth pursuing further because of the serious implications involved, for the existence of a forward-backward asymmetry in π-μ decay would not only necessitate the existence of a non-zero spin for the pion, but also would imply that parity is not conserved in the strong interaction by which the π-meson is produced.

In this experiment the 1 GeV bremsstrahlung beam of the Cornell synchrotron was used to produce π^+ mesons from a liquid hydrogen target. Two exposures were made in which G-5 emulsion stacks were placed behind a 1.5 in. copper absorber at a laboratory angle of 72° in order to observe the decay at rest of mesons which had been produced near resonance at 90° in the center of mass system. By means of magnetic shielding the field at the emulsions was made to be less than the earth's magnetic field. Let $\boldsymbol{k}_1$ be a unit vector in the direction of the bremsstrahlung beam, and let $\boldsymbol{k}_2$, and $\boldsymbol{k}_3$ be unit vectors in the initial direction of the π and μ mesons respectively. Then the μ meson is measured in the co-ordinate system in which $\boldsymbol{y}=\boldsymbol{k}_2$, $\boldsymbol{z}=(\boldsymbol{k}_2\times\boldsymbol{k}_1)$ and $\boldsymbol{x}=\boldsymbol{k}_2\times(\boldsymbol{k}_2\times\boldsymbol{k}_1)$, where the positive directions of $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are respectively designated as right, forward, and up. Fig. 1 gives a schematic representation of this geometry. The two emulsion stacks differed only in orientation. The plane of the emulsion was in the xy plane for stack A and in the yz plane for stack B. By this technique, it is possible to separate experimental biases from real effects in the up-down and left-right directions.

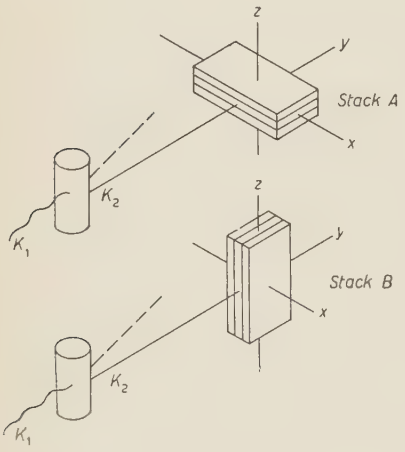

Fig. 1. – Schematic drawing of exposure geometry.

The first 3089 π-μ decays were found by track following: the stacks were scanned along a plane normal to $\boldsymbol{y}$ and mesons crossing this plane were followed to their end and analyzed. No attempt was made to find all of the mesons. This procedure tended to select those mesons which did not suffer large angle scatterings. This method of scanning avoids the biases inherent in area scanning where the events are discovered by recognition of the decay itself, a method for which the efficiency of finding μ mesons whose initial direction is forward or steep is very possibly less than that for finding other events. In order to find the 3089 π-μ decays 3449 mesons were followed to their ends. Of these, 141 formed π^- stars and 219 formed neither π^- stars nor π-μ decays. These 219 endings were checked to make sure that they included only μ mesons and π^- zero prong stars. Approximately the same number

of π-μ decays were found in each of the two stacks. For each decay found by track following, the angles defining the μ direction were measured. The dip measurements were made on Leitz Standard Medical Microscopes under a total magnification of 1600. No unusual procedures were followed other than a displacement of the track to compensate for curvature of field. The distance over which the vertical displacement of the track was measured was left to the discretion of the scanner. The shrinkage factor was estimated by measuring the total thickness of the stack just after exposure.

In addition to the events found by track following 1944 π-μ decays were found in stack *A* by area scanning. For these decays the only information recorded was whether the muon went up or down, right or left, and forward or backward with respect to the initial direction of the pion. An efficiency check showed that the efficiency for finding π-μ decays in the area scanning was greater than 95%.

The data from the track following were statistically analyzed in two ways. First they were tested for an assumed angular distribution of the form $1+a\cos\theta$, where θ is the angle between $\boldsymbol{k}_3$ and the direction of polarization of the π meson. In the absence of any knowledge about this polarization direction, an estimate of the asymmetry parameter a was obtained from the angular distributions with respect to each of the x, y and z directions by means of the relation

$$a_x = 3\,\overline{\cos\theta_x}\,, \quad \text{etc.}$$

The values of these three estimates appear in Table I. Secondly, in order to see whether the data suggest any anisotropy at all and also to check against any experimental biases, a χ^2 test was applied to each of the three histograms of the angular distributions shown in Fig. 2. The χ^2 test yields the probability p that this experimental sample of π-μ decays would look less isotropic than it does if the population from which it was taken were truly isotropic.

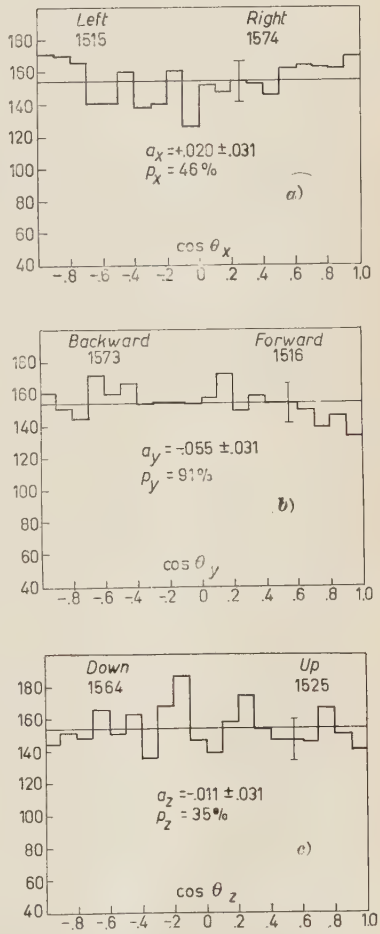

Fig. 2. – Histograms of the π-μ angular distribution.

TABLE I. – *Space components of the π-μ angular distribution.*

	From track following	From area scanning	Total
a_x	$+ .020 \pm .031$	$+ .034 \pm .045$	$+ .025 \pm .026$
a_y	$- .055 \pm .031$	$+ .024 \pm .045$	$- .030 \pm .026$
a_z	$- .011 \pm .031$	$+ .001 \pm .045$	$- .007 \pm .026$

The values of p are shown in Fig. 2. The values of a and p quoted are for the combined data obtained from track following in stacks A and B. The data were also analysed by a χ^2 test by plotting the events in 30 equal sectors of a spherical surface. This test gave $p = 50\%$ in one stack, 40% in the other, and no apparent correlation between the two.

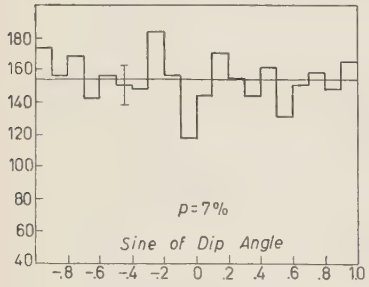

Fig. 3. – Histogram of the combined dip angle distribution.

Analysis of the combined dip angle measurements, shown in Fig. 3, suggests a small experimental bias toward measuring too few events at small dip angles. This is the type of bias one would expect to find as a result of imperfect resolution in the measurement of vertical distances in the emulsion, because the relative error in $\cos\theta$ (which is equal to the sine of the dip angle) is largest near $\cos\theta = 0$. The net effect of a finite resolution is then to decrease the number of events actually observed near $\cos\theta = 0$. No attempt was made to correct the estimates given above for this effect.

From the area scanning also, estimates of the parameter a were obtained by means of $a_x = 2\ (N_F - N_B)/(N_F + N_B)$, etc. These estimates of a as well as the estimates obtained from the combination of all 5 034 events are shown in Table I.

The asymmetry parameter a which is estimated in this experiment is equal to αP where P is the polarization of the π mesons and α is the asymmetry parameter characteristic to the decay itself. We have measured a to a precision of .026 and the result is quite consistent with isotropy. This is the expected result for a spin zero pion. A pion of non-zero spin could produce this result only if it were not appreciably polarized in the production process.

* * *

We wish to thank Dr. KENNETH ROGERS and Dr. PETER STEIN for the opportunity of using their liquid hydrogen target. Our thanks are due also to CAROL SIENKO, LOUISE VAN NEST, and MARY WAKEMAN for most of the microscope work.

RIASSUNTO (*)

È stata fatta l'analisi dei decadimenti di 5034 mesoni π^+ di fotoproduzione arrestati in emulsione nucleare. La distribuzione angolare è isotropica.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

The Mechanism of Neutron Emission in High Energy Fission.

D. M. Skyrme and G. N. Harding

Atomic Energy Research Establishment - Harwell, England

(ricevuto il 16 Giugno 1958)

Summary. — A measurement has been made of the energy spectrum of neutrons evaporated from uranium bombarded with 150 MeV protons, using nuclear emulsion technique. The mean kinetic energy of the neutrons is found to be (2.4 ± 0.2) MeV.

1. – Introduction.

Large numbers of neutrons are known to be emitted in the fission of heavy nuclei by high energy particles. In particular, the fission of uranium and bismuth by 147 MeV protons has been shown [1] to be accompanied by the emission of (13.1 ± 1.8) and (10.0 ± 2.7) neutrons respectively. The mechanism of emission of these neutrons has been studied by a number of authors, the two principal alternative proposals being those of Goeckermann and Perlman [2], who suggest that most of the neutrons are emitted in de-excitation of the nucleus before fission, and of Marquez [3], who advances reasons why it is more likely that fission takes place immediately after capture of the fast particle, the neutrons being subsequently emitted from the moving fragments.

It was pointed out by the latter author that a measurement of correlation between the directions of emission of fission fragments and neutrons would indicate which theory was correct. Such a measurement was therefore made [4] with uranium bombarded by 147 MeV protons, and it was found that the

[1] G. N. Harding: *Proc. Phys. Soc.*, A **69**, 330 (1956).
[2] R. H. Goeckermann and I. Perlman: *Phys. Rev.*, **76**, 628 (1949).
[3] L. Marquez: *Nuovo Cimento*, **12**, 288 (1954).
[4] G. N. Harding and F. J. M. Farley: *Proc. Phys. Soc.*, A **69**, 853 (1956).

ratio of numbers of neutrons detected at 0° and 90° to the direction of the fission fragments was 1.27 ± 0.11. This ratio was shown to be consistent with the emission of the majority of the neutrons before fission, a number between 1.5 and 3.5 being emitted from the moving fragments.

In drawing this conclusion it was necessary to make some assumptions about the energy spectrum of the neutrons emitted from the fragments. No direct evidence was available, but on the basis of theoretical prediction and experimental results in related conditions, a neutron spectrum was assumed having the form $N(E)\,\mathrm{d}E = \alpha^2 E e^{-\alpha E}\cdot \mathrm{d}E$, where E is the kinetic energy of the neutron and α is a constant. The mean neutron energy with the assumed spectrum was 2.67 MeV. MARQUEZ [5] has recently drawn attention to the fact that the anisotropy observed by HARDING and FARLEY would be consistent with the emission from the moving fission fragments of all the neutrons, provided that their mean energy were as high as 6.15 MeV in the laboratory system. To eliminate this remaining source of uncertainty we have now made an experimental determination of the mean energy of the neutrons accompanying the fission of natural uranium by high energy protons.

2. – Experimental procedure and results.

A uranium target was bombarded by the internal proton beam of the Harwell 110 in synchrocyclotron, the mean energy of the protons striking the target being 150 MeV. Neutrons emerging in a small solid angle around 180° to the direction of motion of the incident protons passed through 2 in. diameter holes in concrete blocks and entered a well shielded room 19 metres away. The neutron energy distribution was measured by observing the recoil protons produced in a photographic plate (Ilford C2 emulsion, 400 μm thick). Corrections were applied to take into account the scattering of the neutrons between the target and the scanning zone in the plate.

Subsidiary experiments showed that the background of neutrons arising from sources other than the target was negligible. Since fission occurs in about 80% of the inelastic collisions made by the protons in the target, and the energy distribution of neutrons evaporated from heavy elements which do not undergo fission is very similar to the fission neutron spectrum [6], the error arising from identifying the observed neutrons with those accompanying fission was small.

Neutrons with energy less than 0.6 MeV were not included in the measurements on account of the difficulty of identifying the short proton recoil track.

---

(5) L. MARQUEZ: *Proc. Phys. Soc.*, A **70**, 546 (1957); *Nuovo Cimento*, **5**, 1646 (1957).
(6) D. M. SKYRME and W. S. C. WILLIAMS: *Phil. Mag.*, **42**, 1187 (1951).

Using this low-energy cut-off, correcting in the usual way for the variation with energy of the n-p cross-section and for the loss of protons which leave the emulsion before coming to the end of their range, we obtain a value of (2.81 ± 0.16) MeV for the mean neutron energy. This is clearly an overestimate. When a correction is applied to allow for the presence of neutrons with energy less than 0.6 MeV ([7]), we obtain (2.4 ± 0.2) MeV for the mean neutron energy. The error quoted is a standard deviation based on the number of tracks measured, together with some allowance for uncertainty in the number of neutrons with energy less than 0.6 MeV.

3. – Conclusion.

Our result refers to neutrons emitted both before and after fission has occurred; the mean energy is much smaller than the value necessary to reconcile the observed anisotropy ([4]) with the theory that all the neutrons are emitted after fission. The conclusion drawn by HARDING and FARLEY ([4]) that most of the neutrons are emitted before fission thus appears to be justified.

4. – Note.

Since these measurements were made our attention has been drawn to a similar experiment carried out at a somewhat higher energy by GROSS ([8]). His results are in very satisfactory agreement with those reported here.

([7]) J. M. B. LANG and K. J. LE COUTEUR: *Proc. Phys. Soc.*, A **67**, 586 (1954).
([8]) E. E. GROSS: *UCRL* 3337, *University of California Radiation Laboratory* (1956).

RIASSUNTO (*)

Si è misurato lo spettro d'energia dei neutroni evaporati dall'uranio bombardato con protoni di 150 MeV, servendosi della tecnica delle emulsioni nucleari. Si trova che l'energia cinetica media dei neutroni è (2.4 ± 0.2) MeV.

(*) *Traduzione a cura della Redazione.*

K^- Interactions with Free Protons in Nuclear Emulsion.

R. G. Glasser, N. Seeman and G. A. Snow

Nucleonics Division, U. S. Naval Research Laboratory - Washington 25, D. C.

(ricevuto il 19 Giugno 1958)

Summary. — A total of 19 (K^-, p) elastic scattering events and 14 $K^- + p \rightarrow \Sigma^\pm + \pi^\mp$ events in flight have been found, including the results reported previously. Averaging over the energy region (0 ÷ 82) MeV, the cross sections are respectively, $\sigma_e = (37^{+9}_{-9})$ mb and $\sigma_a = (27^{+8}_{-6})$ mb. The best estimate of the Σ^+/Σ^- ratio at production is $1.2^{+1.0}_{-0.6}$.

A search for K^- meson collisions on free protons in nuclear emulsion has been carried out in the energy region 0 to 82 MeV. A report on the first half of this experiment has been published (1). A description of the emulsion stack, the exposure conditions and the scanning technique is contained in I. The mean energy of the incident K^- meson beam is 81.6 MeV.

The total number of K^- mesons followed is 4 969, corresponding to a total path length of 161 m. Among all the types of K^- interactions with free hydrogen in emulsion, only elastic scattering events ($K^- + p \rightarrow K^- + p$) and absorption events ($K^- + p \rightarrow \Sigma^\pm + \pi^\mp$) can be recognized. The total numbers of elastic and absorption events found in this path lenght are 19 and 14 respectively. Characteristics of the events found since I are listed in Tables I and II. More extensive measurements on event No. 85-35 listed in I as a $K^- + p \rightarrow \Sigma^\pm + \pi^\mp$ event, have shown this assignment to be incorrect (2).

(1) R. G. Glasser, N. Seeman and G. A. Snow: *Nuovo Cimento*, **7**, 142 (1958). (hereafter called I).

(2) The coplanarity condition is well satisfied in this event. However, the velocity of the secondary π-meson from the $\Sigma^\pm$ hyperon decay, as subsequently measured by blob counting 7 cm from the decay point, is incompatible with the velocity deduced from the kinematics of the assigned reaction. Hence this event must be classified as a pseudo hydrogen event arising from a collision of a K^--meson with a proton moving

TABLE I. – *Characteristics of* $K^- + p \to K^- + p$ *events.*

Event	K^- Star (a)	T_{K^-} (b) (MeV)	$\chi^{CM}_{KK'}$ (c)
93-80	(1, 0, 3, 0)	16	114°
41-37	(0, 0, 5, 0)	24	101°
67-51	(0, 0, 3, 0)	25	23°
101-27	(0, 0, 0, 1)	37	98°
42-12	(0, 0, 1, 1)	43	98°
77A-57	(1, 0, 1, 0)	47	128°
52-31	(1, 0, 1, 0)	56	111°
101-39	(1, 0, 1, 0)	63	113°
45B-38	(0, 0, 0, 1)	75	53°
68-38	(0, 0, 4, 0)	83	33°

(*a*) Description of star made by outgoing K^-. The four numbers in parentheses denote the number of light, gray, black, and recoil or electron tracks respectively.

(*b*) T_{K^-} is the energy of the K^--meson at the point of scatter as determined from the kinematics of the event.

(*c*) $\chi^{CM}_{KK'}$ is the angle through which the K^- scatters in the (K^-, p) center of mass system.

TABLE II. – *Characteristics of* $K^- + p \to \Sigma^\pm + \pi^\mp$ *events.*

Event	Hyperon (a)	T_{K^-} (b) (MeV)	$\chi^{CM}_{K^-\pi}$ (c)
43-52 (d)	$\Sigma^\pm \to \pi^\pm + n$ (f)	42	85°
101- 6	Σ^- star (0, 0, 0, 1)	45	94°
51-25	$\Sigma^+ \to p + \pi^0$ (f)	48	72°
65-17	Σ^- star (0, 0, 0, 0)	54	94°
55-12 (d)	$\Sigma^\pm \to \pi^\pm + n$ (f)	55	63°
106- 7 (d)	$\Sigma^\pm \to \pi^\pm + n$ (f)	68	168°
56-22	$\Sigma^\pm \to \pi^\pm + n$ (r)	70	138°

(*a*) The four numbers in parentheses after « Σ^- star » denote the number of light, gray, black, and recoil or electron tracks respectively. (f) or (r) after a decay mode indicates decay in flight or at rest.

(*b*) T_{K^-} is the energy of the K^--meson at the point of interaction as determined from the kinematics of the event.

(*c*) $\chi^{CM}_{K^-\pi}$ is the angle between the K^--meson and the π-meson directions in the (K^-, p) center of mass system.

(*d*) Applying kinematics to these events, a better fit is obtained on the assumption that the hyperon in 43-52 is a Σ^-, that the hyperon in 55-12 is a Σ^+, while no distinction can be made in event 106-7.

inside a heavier nucleus. An examination of the experimental distribution in the angle of non-coplanarity of pseudo hydrogen events indicates that corresponding to the 7 new K^-p absorption events, one might expect $\lesssim 1$ pseudo hydrogen event that satisfies coplanarity. In the second half of this scan, one event was found which satisfied the coplanarity condition but which was ruled out as a K^-p event on the basis of kinematic criteria.

Table III presents σ_e, the cross-section for the reaction $K^- + p \to K^- + p$ and σ_a, the cross-section for the reation $K^- + p \to \Sigma^\pm + \pi^\mp$, in several energy

TABLE III. – *Summary of* (K^-, p) *cross sections for different energy regions.* σ_e *and* σ_a *represent the cross sections for the reactions* $K^- + p \to K^- + p$ *and* $K^- + p \to \Sigma^\pm + \pi^\mp$ *respectively.*

Median (^{a}) energy of K^- (MeV)	Energy interval (MeV)	σ_e (mb)	σ_a (mb)	$\sigma_e + \sigma_a$ (mb)
21	0 — 30	49^{+32}_{-19}	49^{+30}_{-20}	98^{+41}_{-29}
47	30 — 60	41^{+17}_{-12}	35^{+16}_{-11}	76^{+22}_{-17}
72	60 — 82	30^{+13}_{-10}	13^{+10}_{-5}	43^{+16}_{-12}
58	0 — 82	37^{+9}_{-8}	27^{+8}_{-6}	64^{+12}_{-10}

(a) The median energy corresponding to each energy interval is defined as that energy for which there is an equal length of path followed for particles of higher and lower energy.

intervals. Fig. 1 and 2 display the K^- meson kinetic energy and the cosine of the appropriate center of mass angle for each hydrogen event. The energy

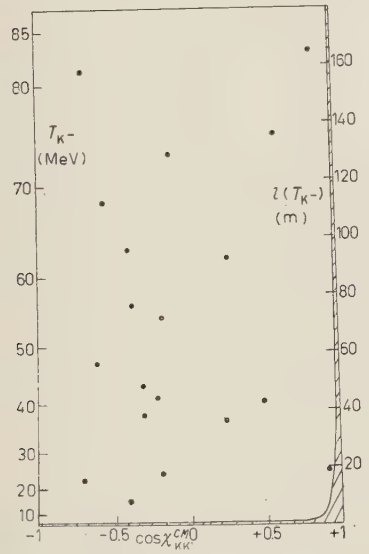

Fig. 1. – (K^-, p) elastic scattering events. $l(T_{K^-})$ is the followed path length of K^--mesons of energy $\leqslant T_{K^-}$. The shaded area denotes the region in which the detection efficiency is low for elastic scatters.

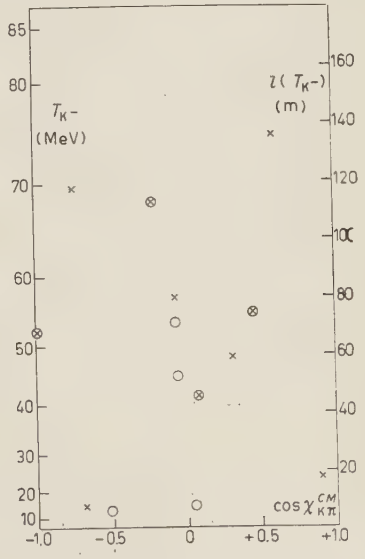

Fig. 2. – (K^-, p) absorption events. $l(T_{K^-})$ is the same as in Fig. 1. The symbols on the graph indicate the reaction products: open circles denote $\Sigma^-\pi^+$, crosses denote $\Sigma^+\pi^-$ crosses in circles denote events which could be either.

scale is chosen in such a way that equal scale lengths denote equal intervals of K^- meson path length followed.

It is clear that only tentative conclusions can be drawn from this limited sample of data. From Table III, it appears that σ_a is a fairly rapidly decreasing function of incident K^- meson energy, while σ_r is comparatively independent of energy. This general trend has been noted previously in a summary of the work of several emulsion groups prepared by M. CECCARELLI [3]. Note that σ_a in the energy interval, $(0 \div 30)$ MeV, in much lower than reported in I.

Some information can be obtained about the Σ^+/Σ^- ratio at production for the absorption reaction. Out of the 14 absorption events, we find 6 Σ^+, 4 Σ^- and 4 $\Sigma^\pm$. This last category consists of Σ hyperons that decay in flight into π mesons that escape our stack. Approximate upper and lower limits are obtained for the Σ^+/Σ^- ratio by assigning the 4 indeterminate events to either of the two charge states in turn. The Σ^+/Σ^- ratio is thus found to lie between 6/8 and 10/4. One can obtain an estimate for the number of $\Sigma^+ \to \pi^+ + n$ decays in flight by combining our data on the observed number of $\Sigma^+ \to p + \pi^0$ decays in flight, which is one, with the known Σ^+ branching ratio $((\Sigma^+ \to p + \pi^0)/(\Sigma^+ \to n + \pi^+) = 0.88 \pm 0.12)$ [4]. The estimate for the Σ^+/Σ^- ratio then becomes 7/7. An independent estimate of the number of Σ^- produced in these 14 (K^-, p) absorption events can be obtained by a somewhat more sophisticated statistical analysis. This method is described in detail in the Appendix. Only experimental information on the Σ^- hyperons that come to rest is used, the essential point being that these are the only Σ hyperons whose sign can unambiguously be determined to be negative. Each of these events is weighted inversely to the probability that it come to rest, that is, proportional to $\exp[+T_i/\tau_-]$, where T_i is the time for the hyperon to come to rest and τ_- is the mean life of the Σ^- hyperon $(\tau_- = 1.67 \cdot 10^{-10}$ s) [5]. This method gives an estimate for the Σ^+/Σ^- ratio of 8/6. Combining the two methods of estimating the Σ^+/Σ^- ratio, with their errors, and including the fluctuation in this ratio for this small sample, which is the major contribution to the final error, one finds that the Σ^+/Σ^- ratio at production is $1.2^{+1.0}_{-0.6}$. This result for (K^-, p) interactions in flight may be compared with the bubble chamber result for (K^-, p) interactions at rest which gives 0.51 ± 0.08 [6].

[3] M. CECCARELLI: *Proc. of the Rochester Conference* (New York, 1957). See also E. LOHRMANN, M. NICOLIĆ, M. SCHNEEBERGER, P. WALOSCHEK and H. WINZELER: *Nuovo Cimento*, **7**, 163 (1958).

[4] G. A. SNOW: *Proc. of the Rochester Conference* (New York, 1957).

[5] M. GELL-MANN and A. ROSENFELD: *Ann. Rev. of Nucl. Sci.* (Palo Alto, California, 1957), p. 411.

[6] L. W. ALVAREZ, H. BRADNER, P. FALK-VAIRANT, J. D. GOW, A. H. ROSENFELD, F. L. SOLMITZ and R. D. TRIPP: *Nuovo Cimento*, **5**, 1026 (1957); and A. H. ROSENFELD: *Report on Rochester Conference* (1957).

* * *

We would like to thank Mrs. J. LEEK and Mrs. G. RONES for their diligent scanning of the plates. We are indebted to Messrs. F. O'DELL and B. STILLER for their advice and aid in the processing of the emulsions, and to Prof. E. LOFGREN and Prof. W. BARKAS and his group for their aid in exposing the stack. Finally we would like to thank Dr. M. M. SHAPIRO for his advice and encouragement.

APPENDIX

In this appendix, the general method which was applied to estimate the number of negative hyperons produced in K^- absorption is discussed.

Consider a process in which an unknown number, N, of particles are emitted. At emission, each particle has a certain probability, p_i, of being detected. If detected the value of p_i can be determined. In a given experiment, n of these particles are detected. Label the particles so that the first n are the detected particles.

Now define for each particle a random variable, ε_i, by:

$$\begin{aligned} \varepsilon_i &= 1 \quad \text{if particle is detected (probability } p_i), \\ &= 0 \quad \text{if particle is not detected (probability } 1-p_i). \end{aligned}$$

The symbol $E(A)$ is used to denote the expected value of a random variable and $V(A)$ its variance. It follows from the definition of ε_i that

$$\begin{aligned} E(\varepsilon_i) &= p_i\,, \\ V(\varepsilon_i) &= E([\varepsilon_i - p_i]^2) = p_i(1-p_i)\,. \end{aligned}$$

We can now define as an estimate of N:

$$\hat{N} = \sum_{i=1}^{N} \frac{\varepsilon_i}{p_i} = \sum_{i=1}^{n} \frac{1}{p_i}\,.$$

This is an unbiased estimate since:

$$E(\hat{N}) = \sum_{i=1}^{N} \frac{E(\varepsilon_i)}{p_i} = N\,.$$

Its variance is given by:

$$V(\hat{N}) = \sum_{i=1}^{N} \frac{V(\varepsilon_i)}{p_i^2} = \sum_{i=1}^{N} \left(\frac{1}{p_i} - 1\right).$$

This expression cannot be directly evaluated since it depends on the values of the p_i for the undetected particles. However it can be estimated by:

$$\hat{V} = \sum_{i=1}^{N} \frac{\varepsilon_i}{p_i^2} - \hat{N} = \sum_{i=1}^{n} \left(\frac{1}{p_i^2} - \frac{1}{p_i} \right).$$

The expected value of $\hat{V}$ is

$$E(\hat{V}) = \sum_{i=1}^{N} \frac{1}{p_i} - N = \sum_{i=1}^{N} \left(\frac{1}{p_i} - 1 \right).$$

That is, $\hat{V}$ is an unbiased estimate of the variance of $\hat{N}$.

In the application discussed in the text, N is the number of negative hyperons produced, a Σ^- is « detected » if it comes to rest, and $p_i = \exp[-T_i/\tau_-]$ is the probability of being « detected ».

RIASSUNTO (*)

Compresi i risultati precedentemente riferiti si sono trovati in totale 19 eventi di scattering elastico (K^-, p) e 14 eventi $K^- + p \to \Sigma^\pm + \pi^\mp$ in volo. Mediando sull'intervallo d'energia $(0 \div 82)$ MeV le sezioni d'urto sono rispettivamente $\sigma_e = (37^{+9}_{-9})$ mb e $\sigma_a = (27^{+8}_{-6})$ mb. Il miglior valore del rapporto Σ^+/Σ^- alla produzione è $1.2^{+1.0}_{-0.6}$.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

Phenomenological Theory of the S Matrix and T, C and P Invariance (II).

J. Werle

Institute for Nuclear Research - University of Warsaw

(ricevuto il 19 Giugno 1958)

Summary. — The paper contains some applications of the general phenomenological theory of the transition matrix which has been given in part I of this work. Besides a general discussion of various physical relations which follow from the T, C and TCP invariance requirements the examples of $K_{\mu 3}$ and hyperon decays are studied in some more detail.

1. – Introduction.

In the first paper of this work (1) we have shown how to construct the most general expressions for the transition matrices M which describe any given reactions between elementary or composite particles of arbitrary spins. It has been assumed that in the S matrix formalism both so called elementary and composite particles are described by local field operators which satisfy just the usual free field equations of motion and free field commutation relations with physical masses. Some justification for such a description has been given, however, without pretending to be a serious proof. Meanwhile Zimmermann (2) has shown on a field theoretical model that in fact in the S matrix formalism one cannot distinguish between elementary and composite particles as all of them can be described by local field operators corresponding to freely moving particles. The same seems to be true even in the case of non-local interactions.

According to its definition the S matrix describes transitions from some

(1) J. Werle: *Nuovo Cimento*, **9**, 569 (1958).

(2) W. Zimmermann: *On the Bound State Problem in Quantum Field Theory* (preprint).

asymptotic states in the infinite past to some final asymptotic states in the infinite future. Therefore, strictly speaking the asymptotic states can consist of freely moving stable particles only. If we want to deal with processes in which unstable particles take part we must consider finite time intervals. However, because of the smallness of the Planck constant we can apply the S matrix formalism also to processes involving unstable particles of not too short lifetimes, *e.g.* particles decaying via weak interactions.

2. – Classification of processes described by the same matrix M.

Consider first a particular reaction, *e.g.*

$$A_1 + A_2 \to B_1 + B_2 \,. \tag{1}$$

Changing one or more of the particles occurring in the initial (final) state into antiparticles occurring in the final (initial) state we obtain a family of closely related processes containing, *e.g.*

$$A_1 \to \bar{A}_2 + B_1 + B_2 \,, \qquad \bar{B}_1 + \bar{B}_2 \to \bar{A}_1 + \bar{A}_2 \,, \quad \text{etc.} \tag{2}$$

The symbols $\bar{A}_i$, $\bar{B}_i$ denote the antiparticles to A_i and B_i respectively. There exists another family of processes which involve just the same particles but nevertheless are quite distinct from those belonging to the first family. This second family can be derived from the process reversed to (1), *i.e.* from

$$B_1 + B_2 \to A_1 + A_2 \,, \tag{3}$$

by the same procedure as that pointed out above. In particular the second family contains what we shall call antiprocess to (1), *i.e.*

$$\bar{A}_1 + \bar{A}_2 \to \bar{B}_1 + \bar{B}_2 \,. \tag{4}$$

It is obvious from the structure of M [1] that both these families of processes are described by the same transition matrix M. However, the matrix M is a sum of two terms

$$M = M_a + M_b \,, \tag{5}$$

where M_a contains a suitable product of field operators whereas M_b contains the corresponding product of their Hermitian adjoints. If the process (1) is described by M_a then all other processes $i_{\mathrm{I}} \to f_{\mathrm{I}}$ which belong to the first family are also described by M_a and all processes $i_{\mathrm{II}} \to f_{\mathrm{II}}$ which belong to

the second family are described by M_b, *i.e.*

$$\langle f_{\mathrm{I}}|M|i_{\mathrm{I}}\rangle = \langle f_{\mathrm{I}}|M_a|i_{\mathrm{I}}\rangle\,, \qquad \langle f_{\mathrm{II}}|M|i_{\mathrm{II}}\rangle = \langle f_{\mathrm{II}}|M_b|i_{\mathrm{II}}\rangle\,. \tag{6}$$

In the phenomenological approach we do not derive the actual form of M from a given Lagrangian or Hamiltonian. Therefore, if we use only the « absolute » invariance requirements [1] there is no relation between the scalar coefficients a_K, a'_K and b_K, b'_K that occur in the general expressions for M_a and M_b respectively. Consequently, there will be no relations between both families of processes either. Definite relations between M_a and M_b can be established either by C or by T invariance requirements or by Hermiticity condition. This can easily be seen from the general form of P, C, T and TCP invariance conditions written in terms of M_a and M_b:

a) $W = TCP$ invariance:

$$WM_aW^{-1} = M_a^+\,, \qquad WM_bW^{-1} = M_b^+\,, \tag{7}$$

b) P invariance:

$$PM_aP^{-1} = M_a\,, \qquad PM_bP^{-1} = M_b\,, \tag{8}$$

c) C invariance:

$$CM_aC^{-1} = M_b\,, \qquad CM_bC^{-1} = M_a\,, \tag{9}$$

d) T invariance:

$$TM_aT^{-1} = M_b^+\,, \qquad TM_bT^{-1} = M_a^+\,, \tag{10}$$

e) Hermitian M:

$$M_a = M_b^+\,, \qquad M_b = M_a^+\,. \tag{11}$$

The asymptotic states we shall be dealing with are assumed to have the form of simple products of one-particle states. For the sake of simplicity we shall furthermore assume that the state of every particle occurring in the initial state $|A\rangle$ or in the final state $|B\rangle$ can be described in a complete manner by the momentum vector $\boldsymbol{p}_k$, the polarization vector $\boldsymbol{s}_k$ [3], and some

[3] In general the state of a freely moving particle of spin S, momentum $\boldsymbol{p}$ and non-vanishing mass is described by $4S$ real parameters. Therefore, a description by means of a polarization unit vector $\boldsymbol{s}$ is possible only for $S = \frac{1}{2}$. However, the formulae given in Sects. **3** and **4** remain valid if one replaces $\boldsymbol{s}$ by the component S_z of the spin vector where the z axis is collinear with $\boldsymbol{p}$.

« charge » quantum numbers ε_k which distinguish a particle from the corresponding antiparticle. Thus we define

$$|A\rangle = \prod_{i=1}^{n_A} a_i^+(\boldsymbol{p}_i, \boldsymbol{s}_i, \varepsilon_i)|0\rangle = |\boldsymbol{p}_A, \boldsymbol{s}_A, \varepsilon_A\rangle , \tag{12}$$

$$|B\rangle = \prod_{j=1}^{n_B} a_j^+(\boldsymbol{p}_j, \boldsymbol{s}_j, \varepsilon_j)|0\rangle = |\boldsymbol{p}_B, \boldsymbol{s}_B, \varepsilon_B\rangle , \tag{13}$$

where $a_k^+(\boldsymbol{p}_k, \boldsymbol{s}_k, \varepsilon_k)$ is the creation operator for the k-th particle in the state defined by $\boldsymbol{p}_k$, $\boldsymbol{s}_k$, ε_k. It follows from the definition of P, C and T operations that

$$\left\{\begin{array}{ll} P|A\rangle = |-\boldsymbol{p}_A, \boldsymbol{s}_A, \varepsilon_A\rangle , & T|A\rangle = |A_T\rangle = |-\boldsymbol{p}_A, -\boldsymbol{s}_A, \varepsilon_A\rangle , \\ C|A\rangle = |\bar{A}\rangle = |\boldsymbol{p}_A, \boldsymbol{s}_A, -\varepsilon_A\rangle , & W|A\rangle = |\bar{A}'\rangle = |\boldsymbol{p}_A, -\boldsymbol{s}_A, -\varepsilon_A\rangle , \end{array}\right. \tag{14}$$

and similarly for the state $|B\rangle$.

Now the process $A \to B$ is described by

$$N_1(\boldsymbol{p}_k, \boldsymbol{s}_k, \varepsilon_k) = |\langle B|M_a|A\rangle|^2 , \tag{15}$$

the corresponding antiprocess $\bar{A} \to \bar{B}$ by

$$N_2(\boldsymbol{p}_k, \boldsymbol{s}_k, \varepsilon_k) = |\langle B|C^{-1}M_bC|A\rangle|^2 , \tag{16}$$

the reverse to the antiprocess, *i.e.* $\bar{B} \to \bar{A}$, by

$$N_3(\boldsymbol{p}_k, \boldsymbol{s}_k, \varepsilon_k) = |\langle A|C^{-1}M_aC|B\rangle|^2 , \tag{17}$$

and the time reversed process $B_T \to A_T$ by

$$N_4(\boldsymbol{p}_k, \boldsymbol{s}_k, \varepsilon_k) = |\langle A|T^{-1}M_bT|B\rangle|^2 . \tag{18}$$

3. – Relations within the same family.

All the processes which belong to the same family are related by the requirement that the matrices M_a and M_b must be invariant under proper Lorentz transformations and are to be constructed from local field operators corresponding to freely moving particles. We shall not discuss here all the possible relations within the same family, confining our attention to those of them which follow from P, C, T or W invariance requirements or Hermiticity condition.

Writing N_3 in the form

$$N_3 = |\langle B|C^{-1}M_a^+C|A\rangle|^2 = |\langle BT^{-1}P^{-1}W^{-1}M_a^+WPT|A\rangle|^2, \tag{19}$$

and using the W invariance condition (7) we obtain

$$N_3 = |\langle B|T^{-1}P^{-1}M_aPT|A\rangle|^2 = |\langle \boldsymbol{p}_B, -\boldsymbol{s}_B, \varepsilon_B|M_a|\boldsymbol{p}_A, -\boldsymbol{s}_A, \varepsilon_A\rangle|^2 . \tag{20}$$

Thus we see that in the case of $W = TCP$ invariance the function N_3 can be obtained from N_1 by a mere change of sign of all the prolarization vectors

$$N_3(\boldsymbol{p}_k, \boldsymbol{s}_k) = N_1(\boldsymbol{p}_k, -\boldsymbol{s}_k) . \tag{21}$$

If M is Hermitian and invariant under C we obtain

$$N_3(\boldsymbol{p}_k, \boldsymbol{s}_k) = N_1(\boldsymbol{p}_k, \boldsymbol{s}_k) . \tag{22}$$

If M is Hermitian and invariant under CP we find that

$$N_3(\boldsymbol{p}_k, \boldsymbol{s}_k) = N_1(-\boldsymbol{p}_k, \boldsymbol{s}_k) . \tag{23}$$

In the case of Hermitian M and T invariance we obtain the following restriction [1]

$$N_R(\boldsymbol{p}_k, \boldsymbol{s}_k) = N_R(-\boldsymbol{p}_k, -\boldsymbol{s}_k) , \tag{24}$$

which means that all the probability distributions must be even functions of $\boldsymbol{p}_k$ and $\boldsymbol{s}_k$. Similar restriction can be found from P invariance requirement which implies that N_R must be even in the momenta $\boldsymbol{p}_k$.

Relations identical in form with (21)–(24) can be found for the second family of processes which are described by M_b.

4. – Relations between different families.

Now we shall investigate some basic relations between processes described by the same M but belonging to different families. Let us start from the problem of particle and antiparticle lifetimes. The lifetime of a particle A

---

[1] J. WRZECIONKO: in print in *Bull. Pol. Acad. Sci.*

which can decay into various channels B is given by the formula

$$\tau_A^{-1} = \sum_B \int |\langle B | M_a | A \rangle|^2 . \tag{25}$$

On the other hand the lifetime $\tau_{\bar{A}}$ of the corresponding antiparticle $\bar{A}$ is given by

$$\tau_{\bar{A}}^{-1} = \sum_B \int |\langle B | C^{-1} M_b C | A \rangle|^2 . \tag{26}$$

The integral signs denote here an integration over all final state momenta and a summation over all final state spins. It is quite obvious from (8), (9) and (14) that $\tau_{\bar{A}} = \tau_A$ if M is invariant under C or CP. In the case of W invariance we find

$$\tau_A^{-1} = \sum_B \int |\langle B | W^{-1} M_a^+ W | A \rangle|^2 = \sum_B \int |\langle \bar{B}' | M_a^+ | \bar{A}' \rangle|^2 . \tag{27}$$

Thus we see that W invariance alone does not lead to any relation between $\tau_{\bar{A}}$ and τ_A. However, if in addition M is Hermitian, *i.e.* if $M_a^+ = M_b$, we obtain again $\tau_{\bar{A}} = \tau_A$. There may be, therefore, three different reasons for the equality of particle and antiparticle lifetimes: 1) C invariance, 2) CP invariance, 3) Hermitian character of M and $W = TCP$ invariance. Experimentally established inequality $\tau_{\bar{A}} \neq \tau_A$ would provide an unequivocal proof that neither C nor CP invariance can be valid. On the other hand nothing definite can be said in this case about TCP invariance unless we know that M is in fact Hermitian.

It is to be mentioned that the proof of an approximate equality of particle and antiparticle lifetimes in the case of particles decaying via weak interaction given in [5] is not quite satisfactory as it is based on the assumption that the final state consists of standing waves only and that there are no strong interactions between the initial and the final states.

Next we shall discuss some relations between probability distributions for a process and the corresponding antiprocess, *i.e.* between N_1 and N_2. It can easily be found that in the case of:

a) C invariance

$$N_2(\boldsymbol{p}_k, \boldsymbol{s}_k) = N_1(\boldsymbol{p}_k, \boldsymbol{s}_k) , \tag{28}$$

b) CP invariance

$$N_2(\boldsymbol{p}_k, \boldsymbol{s}_k) = N_1(-\boldsymbol{p}_k, \boldsymbol{s}_k) , \tag{29}$$

[5] T. D. Lee and C. N. Yang: *Elementary Particles and Weak Interactions.*

c) Hermitian M and W invariance

$$N_2(\boldsymbol{p}_k, \boldsymbol{s}_k) = N_1(\boldsymbol{p}_k, -\boldsymbol{s}_k)\,. \tag{30}$$

Relation (28) can be distinguished from (29) only if parity is not conserved. Relation (29) can be distinguished from (30) by measuring relative signs of terms which are odd in $\boldsymbol{p}_k$ but even in $\boldsymbol{s}_k$ or vice versa. It is to be stressed that if we measure only one independent momentum and one polarization (*e.g.* in π meson and $K_{\mu 2}$ decays) we cannot distinguish between (29) and (30). If we assume that the theory is invariant under W then comparing (28) with (30) we see that the Hermitian character of M and C invariance do not make violation of P invariance impossible but restrict it to terms which are even in the polarization vectors $\boldsymbol{s}_k$. *E.g.* in the case of Hermitian M the occurrence of the term $\boldsymbol{p}_i(\boldsymbol{s}_j \times \boldsymbol{s}_k)$ in the probability distribution would imply violation of both P and T invariance but not that of C invariance.

Relations between a process and the corresponding time reversed one can be obtained from:

a) T invariance:

$$N_4(\boldsymbol{p}_k, \boldsymbol{s}_k) = N_1(\boldsymbol{p}_k, \boldsymbol{s}_k)\,, \tag{31}$$

b) TP invariance:

$$N_4(\boldsymbol{p}_k, \boldsymbol{s}_k) = N_1(-\boldsymbol{p}_k, \boldsymbol{s}_k)\,, \tag{32}$$

c) Hermitian M:

$$N_4(\boldsymbol{p}_k, \boldsymbol{s}_k) = N_1(-\boldsymbol{p}_k, -\boldsymbol{s}_k)\,. \tag{33}$$

Relation (31) can be distinguished from (32) only if parity is not conserved in the process under consideration. Relation (32) can be distinguished from (33) by measuring relative signs of terms which are odd in $\boldsymbol{s}_k$. Relation (31) can be distinguished from (33) by measuring relative signs of terms which are odd in $\boldsymbol{p}_k$, $\boldsymbol{s}_k$.

All the relations given so far in this paper are valid for arbitrary number of interacting particles. Each relation between the squares of the transition matrix elements means some definite relation between angular distributions, polarizations etc., which can easily be found for any particular case from the general formulae given above.

5. – Static properties of particles and antiparticles.

It follows from (7) that it is not possible to obtain a relation between $A \to B$ and $\bar{A} \to \bar{B}$ if one uses only the W invariance condition. The authors who claim to have obtained such relations use in fact (more or less explicitly)

some additional restrictions *e.g.* Hermiticity condition (6). It is true, however, that the W invariance of the theory gives definite relations between several « static » characteristics of stable particles and antiparticles. In fact the equations of motion for the field operators φ_A^r and ψ_A^u which occur in M contain physical (renormalized) masses of particles. The definition of the C transformation given in (1) implies strict equality of particle and antiparticle masses. Another definition of particle-antiparticle conjugation allowing the masses of particles and antiparticles to be different would inevitably destroy the W invariance of the theory. On the other hand if one has a theory described by a Hamiltonian which is invariant under W one can easily show that the mass of a stable particle must be the same as the mass of the corresponding antiparticle, the magnetic moments must have the same absolute magnitudes but opposite signs etc. (5). The same relations obtain with high accuracy in the case of unstable particles decaying via weak interaction.

6. – Example: $K_{\mu 3}$ and K_{e3} decays.

In the present section we shall investigate the $K_{\mu 3}$ and K_{e3} decays which are both interesting and simple enough to give an illustration of some uses of the explicit form of the transition matrix given in (1,7). We assume that the K-mesons which decay according to the scheme

$$K \to \pi + \nu + \mu(e) \tag{34}$$

have no spin. Thus the process (34) belongs to the type $(0, 0, \frac{1}{2}, \frac{1}{2})$ and the corresponding transition matrix has the following general form

$$M = \int \delta_{(4)}(p_1 - p_2 - p_3 + p_4) \prod_{L=1}^{4} \{\delta(p_L^2 - m_L^2)\, \mathrm{d}p_L\} \cdot \cdot \{\varphi_1 \varphi_2^+ \bar{\psi}_3 \Gamma \psi_4 + \bar{\psi}_4 \widehat{\Gamma} \psi_3 \varphi_2 \varphi_1^+\}, \tag{35}$$

with

$$\begin{cases} \Gamma = a_s + \gamma_5 a_s' + \dfrac{1}{m_1}(\gamma p_1)(a_v + \gamma_5 a_v'), \\ \widehat{\Gamma} = b_s - \gamma_5 b_s' + (b_v - \gamma_5 b_v')(\gamma p_1)\dfrac{1}{m_1}. \end{cases} \tag{36}$$

(6) Compare *e.g.* ref. (5) and: T. KINOSHITA and A. SIRLIN: *Phys. Rev.*, **108**, 844 (1957); T. D. LEE, R. OEHME and C. N. YANG: *Phys. Rev.*, **106**, 340 (1957); J. J. SAKURAI: *Phys. Rev.*, **109**, 980 (1958).

(7) J. WERLE: in print in *Bull. Pol. Acad. Sci.*

The symbols $\varphi_1(p_1)$, $\varphi_2(p_2)$, $\psi_3(p_3)$, $\psi_4(p_4)$ denote the field operators of the K-meson, π-meson, μ-meson (or electron) and neutrino respectively. The scalar coefficients a_s, b_s etc., are in general some unknown functions of two independent scalar products of the momenta. The meaning of other symbols is the same as in [1].

For the sake of convenience consider first a K^- decaying at rest. We shall assume that the neutrino which is emitted in K^- decay is completely polarized in the direction antiparallel to its direction of flight. Denoting the part of the transition matrix which describes K^- decay by M_a we can express our assumption about the neutrino in the following way:

$$a_s' = -a_s\,, \qquad a_v' = -a_v\,. \tag{37}$$

Because of (37) the number of independent scalar coefficients is now reduced to two. We shall take the plane of decay to be the (x, y) plane with the μ-meson (electron) momentum $\boldsymbol{p}_3$ pointing in the direction of the x axis. Straightforward calculation of the square of the transition matrix element [8] gives

$$N = Q(F + R\,\boldsymbol{sp} + U\,\boldsymbol{sq} + V\,\boldsymbol{s}(\boldsymbol{p}\times\boldsymbol{q})) = Q(F + s_x X + s_y Y + s_z Z)\,, \tag{38}$$

where

$$\left\{\begin{aligned} F &= a_s^* a_s(E - p\alpha) + a_v^* a_v(E + p\alpha) + (a_s a_v^* + a_s^* a_v)m\,,\\ X &= a_s^* a_s(E\alpha - p) + a_v^* a_v(E\alpha + p) + (a_s a_v^* + a_s^* a_v)m\alpha\,,\\ Y &= [(a_s^* a_s + a_v^* a_v)m + (a_s\, a_v^* + a_s^* a_v)E]\beta\,,\\ Z &= i(a_s^* a_v - a_s a_v^*)p\beta\,. \end{aligned}\right. \tag{39}$$

The unit vector $\boldsymbol{s}$ denotes the direction of measurement of the μ-meson (electron) polarization. To simplify notation we have put $\boldsymbol{p}_3 = \boldsymbol{p}$, $\boldsymbol{p}_4 = \boldsymbol{q}$. The symbols E and m denote the energy and mass of the μ-meson (electron); $\alpha = \cos\vartheta$, $\beta = \sin\vartheta$ where ϑ is the angle between $\boldsymbol{p}$ and $\boldsymbol{q}$. The quantities Q, R, U, V, F are all even functions of the momenta. Therefore X and Y are odd but Z is even in the momenta.

It can be easily verified that the relation

$$F^2 = X^2 + Y^2 + Z^2 \tag{40}$$

is satisfied identically, *i.e.* for arbitrary a_s and a_v. This means that the μ-mesons (electrons) emitted in $K_{\mu 3}$ (K_{e3}) decays are completely polarized

---

[8] Covariant polarisation formalism proposed by L. Michel and A. S. Wightman (*Phys. Rev.*, **98**, 1190 (1955)) is used here.

provided that the neutrinos are completely polarized. Actual direction of the μ-meson (electron) polarization vector $\boldsymbol{\zeta}$ depends of course on the correlation parameters E and ϑ and is given by the formulae (9)

$$(41)\qquad \begin{cases} \zeta_x = \dfrac{X}{F} = \dfrac{(E\alpha - p) + cc^*(E\alpha + p) + (c^* + c)m\alpha}{(E - p\alpha) + cc^*(E + p\alpha) + (c^* + c)m}, \\ \zeta_y = \dfrac{Y}{F} = \dfrac{[(1 + cc^*)m + (c^* + c)E]\beta}{(E - p\alpha) + cc^*(E + p\alpha) + (c^* + c)m}, \\ \zeta_z = \dfrac{Z}{F} = \dfrac{i(c - c^*)p\beta}{(E - p\alpha) + cc^*(E + p\alpha) + (c^* + c)m}, \end{cases}$$

where $c = a_v/a_s$. The polarization vector $\boldsymbol{\zeta}$ depends on c in a quite sensitive manner which can be illustrated by the following numerical examples calculated for the particular case $\vartheta = \pi/2$:

c	ζ_x	ζ_y	ζ_z
0	$-p/E$	m/E	0
-1	0	1	0
-1	0	-1	0
i	0	m/E	$-p/E$
$-i$	0	m/E	p/E
$\dfrac{1+i}{\sqrt{2}}$	0	$\dfrac{E+\sqrt{2}m}{\sqrt{2}E+m}$	$\dfrac{-p}{\sqrt{2}E+m}$
$-\dfrac{1+i}{\sqrt{2}}$	0	$-\dfrac{E+\sqrt{2}m}{\sqrt{2}E+m}$	$\dfrac{p}{\sqrt{2}E+m}$
∞	p/E	m/E	0

Having the experimental value of $\boldsymbol{\zeta}$ as a function of E ane ϑ one can find the corresponding value of c simply by solving (41) with respect to c. Since the unit vector $\boldsymbol{\zeta}$ has two independent components we obtain in this way two independent equations from which the real and imaginary parts of c can be calculated. Once we know c we can obtain the value of $|a_s(E, \vartheta)|$ from pure correlation measurements. In fact apart from a known phase space factor the angular correlations are described by

$$(42)\qquad F = a_s a_s^* \left[(E - p\alpha) + cc^*(E + p\alpha) + (c^* + c)m\right].$$

(9) The exact form of the polarization vector $\boldsymbol{\zeta}$ for some particular « pure » couplings has been calculated in J. WERLE: *Nucl. Phys.*, **6**, 1 (1958).

Therefore, if we know $c(E, \vartheta)$ and the experimental form of the angular correlation function we can easily calculate $|a_s(E, \vartheta)|^2$. Thus we see that the coefficients a_s, a_v can be determined up to a common phase factor of modulus unity by combined polarization and correlation measurements.

If the assumption about the complete polarization of the neutrino turns out to be not true we have to calculate N_1 using the most general form of Γ given by (36). Because of the vanishing mass of the neutrino this can be done by simple replacement of $a_i a_j^*$ by: $a)$ $\frac{1}{2}(a_i a_j^* + a_j' a_i'^*)$ in F and Z; $b)$ $-\frac{1}{2}(a_i' a_j^* + a_j' a_i^*)$ in X and Y.

Next let us consider some relations between K^- and K^+ decays which are imposed by C, T invariance requirements or Hermitian character of M. According to [1] we have the following conditions for:

a) C invariance:

$$\frac{b_s}{a_s} = \frac{b_v}{a_v} = \frac{-b_s'}{a_s'} = \frac{-b_v'}{a_v'} = \varrho_C\,, \tag{43}$$

b) T invariance:

$$\frac{b_s}{a_s} = \frac{b_v}{a_v} = \frac{b_s'}{a_s'} = \frac{b_v'}{a_v'} = \varrho_T^*\,, \tag{44}$$

c) Hermitian M:

$$\frac{b_s}{a_s^*} = \frac{b_v}{a_v^*} = \frac{b_s'}{a_s'^*} = \frac{b_v'}{a_v'^*} = 1\,, \tag{45}$$

where ϱ_C and ϱ_T are constant transformation phases of modulus unity. It is worth-while mentioning that in spite of the assumption (37) we can still satisfy either C or T invariance requirements. In fact putting the irrelevant phase factors $\varrho_C = \varrho_T = 1$ we see that both conditions (37) and (43) are satisfied by

$$b_s = b_s' = -a_s' = a_s\,, \qquad b_v = b_v' = -a_v' = a_v\,, \tag{46}$$

whereas (37) and (44) are satisfied by

$$b_s = -b_s' = -a_s' = a_s\,, \qquad b_v = -b_v' = -a_v' = a_v\,. \tag{47}$$

It can easily be seen that the condition (45) is incompatible with (46) but can be made compatible with (47). In other words the complete longitudinal polarization of the neutrino and the requirement of C invariance can be satisfied by a non-Hermitian transition matrix only.

Denoting the polarization vectors of μ^- and μ^+ (or e^- and e^+) emitted whit

the same energies and at the same angles in K^- and K^+ decays by ζ^- and ζ^+ respectively we can easily find from (28)–(30) the following relations ([10]).

a) *C* invariance:

$$\zeta^+ = \zeta^- , \tag{48}$$

b) *CP* invariance:

$$\zeta_x^+ = -\zeta_x^- , \qquad \zeta_y^+ = -\zeta_y^- , \qquad \zeta_z^+ = \zeta_z^- , \tag{49}$$

c) Hermitian *M* and *W* invariance:

$$\zeta^+ = -\zeta^- , \tag{50}$$

d) Hermitian *M* and *T* invariance:

$$\zeta_z^+ = \zeta_z^- = 0 . \tag{51}$$

If the neutrinos are not completely polarized the μ-mesons and electrons emitted in $K_{\mu 3}$ and K_{e3} decays will be also only partially polarized and the absolute magnitudes of the polarization vectors ζ^+ and ζ^- will be smaller than unity.

The analysis of the actual form of *M* can be largely simplified if one observes that the unitarity condition $SS^+ = 1$ implies that the matrix *M* which describes $K_{\mu 3}$ and K_{e3} decays is in fact Hermitian. Strictly speaking we find that the ratio of the imaginary to the real part of *M* is neglegible since it is of the order of magnitude of the scattering amplitudes for the processes $\pi^0 + \mu(e) \to \pi^0 + \mu(e)$, $\nu + \mu(e) \to \nu + \mu(e)$ etc. However, in the case of K^0 decays the corresponding ratio can amount to a few percent.

If *M* is Hermitian the possibility of *C* invariance is ruled out by the observation of the longitudinal polarization of the μ-mesons (electrons). On the other hand the *T* invariance can be tested by measuring ζ_z: If one finds $\zeta_z \neq 0$ violation of *T* invariance would be proved. The validity of the *TCP* theorem can then be tested by checking the relation (50). However, it is to be noted that $\zeta_z = 0$ not only in the case of Hermitian and *T* invariant *M* but also in the more general case of real a_v/a_s and a_v'/a_s'.

If one neglects all the final state interactions one can simplify somewhat the last stage of the phenomenological analysis of the actual form of *M* by introducing three independent coefficients which are functions of only one kinematic parameter ([11]) (*e.g.* π-meson energy).

([10]) Compare: J. WERLE: *Nucl. Phys.*, **4**, 171 (1957); H. STAPP: *Phys. Rev.*, **107**. 634 (1957); J. J. SAKURAI: *Phys. Rev.*, **109**, 980 (1958).

([11]) See ref. ([9]) and ([10]) and: S. FURUICHI *et al.*: *Proc. Theor. Phys.*, **16**, 4 (1956); A. PAIS and S. B. TREIMAN: *Phys. Rev.*, **105**, 1616 (1957); S. W. MCDOWELL: *Nuovo Cimento*, **6**, 1445 (1957).

7. – Example: Hyperon decays.

Let us next consider hyperon decays which provide another simple example of some applications of the phenomenological theory of the transition matrix. Assuming that the hyperons Λ^0 and Σ which decay into a nucleon and a π-meson have spin $\frac{1}{2}$ we can write for them the following general expression for the transition matrix

$$M = \int \delta_{(4)}(p_1 - p_2 - p_3) \prod_{L=1}^{3} \{\delta(p_L^2 - m_L^2)\, \mathrm{d}p_L\} \cdot \{\varphi_3^+ \overline{\psi}_2(a + \gamma_5 a')\psi_1 + \overline{\psi}_1(b - \gamma_5 b')\psi_2 \varphi_3\}, \tag{52}$$

where the symbols $\psi_1(p_1)$, $\psi_2(p_2)$, $\varphi_3(p_3)$ denote the field operators of the hyperon, nucleon and π-meson respectively. The coefficients a, a', b, b' are unknown complex constants. The first part of (52) describes hyperon decays whereas the second part describes antihyperon decays.

Let us consider a hyperon of polarization $\boldsymbol{\zeta}$ decaying at rest into a nucleon of momentum $\boldsymbol{p}$ and polarization $\boldsymbol{s}$ and a π-meson of momentum $-\boldsymbol{p}$. Straightforward calculation leads to the following expression for the probability function

$$N = Q\,[A(1 + (\boldsymbol{s}\boldsymbol{p}_0)(\boldsymbol{\zeta}\boldsymbol{p}_0)) + B\,\boldsymbol{s}(\boldsymbol{p}_0 \times (\boldsymbol{\zeta} \times \boldsymbol{p}_0)) + C(\boldsymbol{\zeta}\boldsymbol{p}_0 + \boldsymbol{s}\boldsymbol{p}_0) + D\,\boldsymbol{s}(\boldsymbol{\zeta} \times \boldsymbol{p}_0)]\,, \tag{53}$$

with [12]

$$\begin{cases} A = (E + m)aa^* + (E - m)a'a'^*\,, \\ B = (E + m)aa^* - (E - m)a'a'^*\,, \\ C = p(a'a^* + aa'^*)\,, \\ D = pi(aa'^* - a'a^*)\,. \end{cases} \tag{54}$$

The symbols m and E denote the mass and energy of the nucleon; $\boldsymbol{p}_0 = \boldsymbol{p}/p$. The angular distribution of the nucleons is given by the function $(1 + (C/A)\boldsymbol{\zeta}\boldsymbol{p}_0)$ which can be obtained from (53) by putting $\boldsymbol{s} = 0$. If we take the $(\boldsymbol{\zeta}, \boldsymbol{p})$ plane as (x, y) plane with the vector $\boldsymbol{p}$ pointing in the direction of the x axis we find

---

[12] The formulae (53)-(55) are equivalent to those given by T. D. LEE and C. N. YANG, and R. SPITZER and H. P. STAPP in their partial wave analysis of hyperon decays. See: T. D. LEE, J. STEINBERGER, G. FEINBERG, P. K. KABIR and C. N. YANG: *Phys. Rev.*, **106**, 1367 (1957); T. D. LEE and C. N. YANG: *Phys. Rev.*, **108**, 1645 (1957); R. SPITZER and H. P. STAPP: *Phys. Rev.*, **109**, 540 (1958).

from (53) the following expression for the nucleon polarization vector $\boldsymbol{\zeta}'$:

$$\zeta'_x = \frac{A\zeta_x + C}{A + C\zeta_x}, \qquad \zeta'_y = \frac{B\zeta_y}{A + C\zeta_x}, \qquad \zeta'_z = \frac{-D\zeta_y}{A + C\zeta_x}. \tag{55}$$

One can determine the absolute magnitude of a and the complex ratio a'/a by measuring: 1) the lifetime of the hyperon ($\tau \sim A$), 2) the asymmetry parameter in the angular distribution, 3) the polarization of the nucleons emitted in some suitably chosen direction. The simplest situation obtains if one measures the polarization of nucleons emitted in the plane of production of the hyperons because then $\zeta_x = 0$.

It can easily be seen from (54) that the maximum possible absolute value $|\boldsymbol{\zeta}|$ of the asymmetry parameter is obtained if $a'/a = \sqrt{(E+m)}/(E-m)$. It is interesting to note that all the decay nucleons will then be completely longitudinally polarized irrespective of the value of the hyperon polarization vector $\boldsymbol{\zeta}$ ([13]).

Consider next the problem of C and CP (or T) invariance. The tests for C and CP invariance which have been given in a general form in Sect. 4 require extensive knowledge of both hyperon and antihyperon decays. Thus we find that both C and CP invariance lead to equal lifetimes for the hyperon and the corresponding antihyperon. One can distinguish between C and CP invariance comparing the angular distributions. In the case of C invariance the asymmetry parameters in hyperon and antihyperon decays must be equal whereas in the case of CP invariance the asymmetry parameters must have opposite values. The polarization vectors of the nucleons and antinucleons are equal in the case of C invariance, but in the case of CP invariance one must change in (53) the signs of terms which are odd in the momentum $\boldsymbol{p}$ (*i.e.* of C and D). It is worth-while stressing that one obtains opposite asymmetry parameters for hyperon and antihyperon decays also in the case of Hermitian M and TCP invariance. We can distinguish between this case and that of CP invariance only by measuring the relative signs of ζ_z in hyperon and antihyperon decays. However, because of the presence of strong interactions in the final state there is rather little chance for M to be Hermitian.

So far we do not know anything about the antihyperons and even if they will be very soon discovered the measurements of their properties may turn out to be much more difficult than in the case of hyperons. It seems, therefore, advisable to find another method for testing C and CP (or T) invariance which will not require so much knowledge about antihyperons. We shall propose here an indirect method based on the unitarity of the S matrix. The unitarity condition $SS^{+} = 1$ relates the hyperon and antihyperon decays with

([13]) J. SZYMANSKI: *Bull. Pol. Acad. Sci.*, **5**, 893 (1957).

pion-nucleon scattering giving a possibility of replacing all the antihyperon data by some data about pion-nucleon scattering.

Putting $S = 1 + iM + iR$ we can write the unitarity condition in the following form

$$i(M - M^+) + RM^+ + RR^+ = -i(R - R^+) - MM^+ - MR^+ . \tag{56}$$

For the sake of simplicity we shall consider here a single channel decay, *e.g.* $\Sigma^- \to \mathrm{n} + \pi^-$. The matrix M is then given by (52) and R describes all other processes except Σ^- decay. Let us calculate the matrix elements of (56) which correspond to the transition under consideration. It can easily be seen that the matrix elements of the right hand side of (56) vanish identically. The last term on the left hand side gives a very small contribution of the order g^2 which can be neglected when compared with the first two terms which are both of the first order in the weak coupling constant g. Thus neglecting the terms which are of higher order in g we find

$$i\langle \mathrm{n}, \pi \,|\, M - M^+ \,|\, \Sigma \rangle = -\int \langle \mathrm{n}, \pi \,|\, R \,|\, \mathrm{n}', \pi' \rangle \, \langle \mathrm{n}', \pi' \,|\, M^+ \,|\, \Sigma \rangle \, \mathrm{d}n' \, \mathrm{d}\pi' . \tag{57}$$

The integral sign denotes here an integration over the momenta and a summation over the spins of the « intermediate » neutron and π-meson. Since each of the matrix elements on the right hand side of (57) contains a four-dimensional δ-function, four integrations can be carried out explicitly. There remains an integration over the angles which can be carried out if we know the probability amplitude for pion-neutron scattering at one energy value fixed by the masses of Σ^-, n and π^- (~ 117 MeV). Evaluating (57) we obtain the following set of equations

$$a - b^* = (\lambda - 1) b^* , \qquad a' - b'^* = (\mu - 1) b'^* . \tag{58}$$

If we assume that the probability amplitude for pion-neutron scattering is known then the constants λ and μ are also known. It follows from (58) that

$$\frac{a'}{a} = \frac{\mu b'^*}{\lambda b^*} . \tag{59}$$

Now according to [1] the T and C invariance conditions can be written in the form

$$\frac{b'}{b} = \pm \frac{a'}{a} , \qquad \left| \frac{a}{b} \right| = 1 , \tag{60}$$

where the (+) sign refers to the case of T invariance, the (—) sign to that of C invariance. First we observe that if either of the relations is to be satisfied the numbers μ and λ must be of modulus unity, *i.e.*

$$\mu = \exp[i\delta'], \qquad \lambda = \exp[i\delta]. \tag{61}$$

Inserting (60) and (61) into (59) we obtain the following form for the T and C invariance conditions:

$$\exp[2i\eta] = \pm \exp[i(\delta' - \delta)], \tag{62}$$

where η is the relative phase between a' and a (*i.e.* $a'/a = |a'/a| \exp[i\eta]$). Thus if we know the probability amplitude for pion-nucleon scattering [14] and the polarization of the neutrons emitted in Σ^- decay we can easily check whether (61) is in fact satisfied and, if so, what choice of sign can satisfy (62).

* * *

The author would like to express his thanks to Prof. L. Infeld and to other members of the Physical Faculty of the Warsaw University for their stimulating interest in the work and many helpful discussions.

[14] According to Lee *et al.* (see ref. [12]) $\delta' - \delta = 2(\delta_p - \delta_s)$ where δ_s and δ_p are the phase shifts of $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ waves of $(\pi^- + n)$ scattering at 117 MeV.

RIASSUNTO (*)

Il lavoro contiene alcune applicazioni della teoria fenomenologica generale della matrice di transizione data nella parte I del presente lavoro. Oltre alla discussione generale di varie relazioni fisiche che conseguono dalle esigenze dell'invarianza T, C e TCP, si studiano con maggior dettaglio gli esempi del decadimento $K_{\mu 3}$ e degli iperoni.

(*) *Traduzione a cura della Redazione.*

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori)

The Mesic Coupling Constants from Scattering and Photoproduction Experiments.

L. Bertocchi and A. Minguzzi

Istituto di Fisica dell'Università - Bologna
Istituto Nazionale di Fisica Nucleare - Sezione di Bologna

(ricevuto il 18 Luglio 1958)

The π-N scattering dispersion relations can be used as a tool for defining the renormalized meson-nucleon coupling constant, the coupling constant being defined by the ratio [1]

$$\frac{\langle p' | J(0) | p \rangle}{\bar{u}(p')\gamma_5 u(p)} = g\left((p'-p)^2\right), \tag{1}$$

taken at the point $(p'-p)^2 = -\mu^2$; $J(0)$ is the meson current operator, g is a real, invariant function of $(p'-p)^2$.

Even the photoproduction dispersion relations [2], when the photon field in the unphysical range is neglected, allow to define the mesic coupling constant and this constant must be a priori identical to the one defined by the π-N scattering dispersion relations.

Actually it is impossible to analyse the photoproduction experiments in terms of dispersion relations and one defines the mesic coupling constant from photoproduction experiments at the threshold using the Kroll-Ruderman theorem [3].

In order to establish the relation between the two definitions of the mesic coupling constant it is more convenient to use the rederivation of the Kroll-Ruderman theorem in the Heisenberg representation, as done by Low [4]; and the threshold S-matrix amplitude for charged meson photoproduction allows the definition of the mesic coupling constant through (1), taken at the point $(p'-p)^2 = 0$.

This definition is a consequence of the essential hypothesis, used in order to establish the Kroll-Ruderman theorem, that the mass of the « real » final meson vanishes.

(1) Bogoljubov, Medvedev and Polivanov: *Lecture notes.*
(2) A. A. Logunov, A. N. Tavkhelidze and L. D. Solovyov: *Nucl. Phys.*, **4**, 427 (1957).
(3) N. M. Kroll and M. A. Ruderman: *Phys. Rev.*, **93**, 233 (1954).
(4) F. E. Low: *Phys. Rev.*, **97**, 1391 (1954).

We have tried to examine what is the difference between the two definitions of the mesic constant; clearly in the fixed source limit this difference vanishes.

This is no more the case in the relativistic treatment, and we have used a perturbative approach in order to estimate this difference.

Actually nothing better can be done: more reliable calculations based on spectral representations of (1) [5]

$$F(q^2) = \frac{1}{\pi}\int_{(3\mu)^2}^{\infty} \mathrm{d}(m^2)\,\frac{\varrho(m^2)}{(m^2-q^2)m^{2N}} + \sum_{n=0}^{N-1} c_n q^{2n}\,, \tag{2}$$

are impossible even if a number of simplifying hypothesis are done [6].

The spectral function $\varrho(m^2)$ can be written as a sum of terms proportional to

$$\langle p'|\overline{\eta}(0)n\rangle\langle n|J^{+}(0)|0\rangle\,, \tag{3}$$

where $\overline{\eta}(0)$ and $J^{+}(0)$ are respectively the nucleon and meson currents: the summation is performed over all the states $|n\rangle$ which are strongly coupled to the meson-nucleon fields.

Even if the sum is severely truncated by keeping only the lowest mass states, *i.e.* the $|n\rangle$ states are only 3π-states, (3) reduces to a product of a S-matrix element corresponding to the

$$\mathrm{N}+\overline{\mathrm{N}}\to\pi+\pi+\pi \qquad \text{or} \qquad \mathrm{N}+\pi\to\mathrm{N}+\pi+\pi\,,$$

processes, multiplied by the S-matrix element corresponding to the process

$$\pi+\pi\to\pi+\pi\,.$$

But these matrix elements actually can be known only in the Born approximation; so it is simpler to apply the perturbative method directly to (1).

The matrix element $\langle p'|J(0)|p\rangle$ can be written (we neglect the trivial dependence on the isotopic spin index)

$$\langle p'|J(0)|p\rangle = \left(\varphi_{p'}\,\Big|\,\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int \mathrm{d}t_1\cdot\mathrm{d}t_2\ldots\mathrm{d}t_n\,p[\boldsymbol{H}_I(t_1)\cdot\boldsymbol{H}_I(t_2)\ldots\boldsymbol{H}_I(t_n)\cdot\boldsymbol{J}(0)]\,\Big|\,\varphi_p\right). \tag{4}$$

where $\boldsymbol{J}(0)$ is the meson current in the interaction representation and $\boldsymbol{H}_I(t_j)$ is the total Hamiltonian (in the interaction representation).

Introducing an external meson potential $\varphi^{\mathrm{ext}}(x)=\delta^{(4)}(x)$ (4) can be rewritten [7]

$$\langle p'|J(0)|p\rangle = \left(\varphi_{p'}\,\Big|\,\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int \mathrm{d}t_1\cdot\mathrm{d}t_2\ldots\mathrm{d}t_n\int \mathrm{d}t\cdot p[\boldsymbol{H}_I(t_1)\cdot\boldsymbol{H}_I(t_2)\ldots\boldsymbol{H}_I(t_n)\cdot\boldsymbol{H}_I^{\mathrm{ext}}(t)]\,\Big|\,\varphi_p\right) \tag{5}$$

[5] H. J. BREMERMANN, R. OEHME and J. G. TAYLOR: *Phys. Rev.*, **109**, 2178 (1958). We assume that this formula is valid for the experimental value of the ratio μ/M.

[6] G. F. CHEW, R. KARPLUS, S. GASIOROWICZ and F. ZACHARIASEN: *Phys. Rev.*, **110**, 265 (1958).

[7] For this technique see: L. BERTOCCHI and A. MINGUZZI: *Nuovo Cimento*, **8**, 783 (1958), in which the same notations are employed.

and this is the Feynman amplitude for the scattering of a nucleon by the external meson field.

We have calculated (5) up to the second order, and the renormalization prescription is the same as used by KROLL and RUDERMAN, namely

$$\lim_{p' \to p} \frac{\bar{u}(p')\Gamma_5(p', p)u(p)}{\bar{u}(p')\gamma_5 u(p)} = 1 .$$

From this it can be shown that the renormalized coupling g_r coincides neither with $g(-\mu^2)$ nor with $g(0)$; the relation between the three coupling constants in this order, correct up to $(\mu/M)^2$ terms, is

$$g(-\mu^2) = g_r\left(1 - \frac{1}{2\pi} g_r^2 \left(\frac{\mu}{M}\right)^2\right),$$

$$g(0) = g_r\left(1 + \frac{0.04}{\pi} g_r^2 \left(\frac{\mu}{M}\right)^2\right).$$

So the difference between $g(-\mu^2)$ and $g(0)$ is of the order of $(\mu/M)^2$.

This feature perhaps is valid in every order of perturbation.

Now the threshold photoproduction amplitudes used in order to define $g(0)$ are correct only up to μ/M terms; so we can think we have lumped the difference between $g(-\mu^2)$ and $g(0)$ into the neglected higher order in μ/M terms in the photoproduction amplitudes, and in this sense the two coupling constants are the same.

On the Mass Measurements in G-5 and K-5 Emulsions.

L. Ciuffolotti, G. Luzzatto and G. Tomasini

Istituto di Fisica dell'Università - Genova
Istituto Nazionale di Fisica Nucleare - Sezione di Genova

G. Cortini

Istituto di Fisica dell'Università - Catania
Centro Siciliano di Fisica Nucleare - Catania

(ricevuto il 22 Luglio 1958)

A series of ionization measurements has been made in order to make a first analysis of the new K-5 Ilford emulsions.

The apparatus employed has been already described by De Marco *et al.* [1].

A total of 2 480 cells of 40 μm length were measured. The K-5 plates had been exposed to 150 MeV protons and to 38 MeV mesons from the Liverpool machine.

The results are shown in Fig. 1, where the different parameters are plotted against the variable R/M, *i. e.* residual range over mass expressed in mm/proton masses. The measured quantities are the following:

1) $100\,B$ = total number of gaps per 100 μm cell;
2) $100\,H(r)$ = total number per cell of gaps longer than r;
3) $100\,l$ = total length of gaps per cell;
4) w = mean gap length of all gaps.

Fig. 2 shows some of the present results in comparison with those obtained previously [2] with G-5 plates rather underdeveloped (stack A), and with those obtained by Castagnoli *et al.* [3] by means of G-5 plates very strongly developed (stack B).

We propose to study the effectiveness of mass evaluations obtained by means of the range-ionization measurements.

Therefore we define a « discrimination parameter »

$$\chi_f = \delta M_f(100)/M\,, \tag{5}$$

where M is the mass of a particle whose track has a known residual range, and $\delta M_f(100)$ is the standard fluctuation of the M values obtained by measuring a parameter f in a given plate, at a given residual range, along a standard 100 μm cell. f may be any of the quantities (1) to (4).

---

[1] A. De Marco, R. Sanna and G. Tomasini: *Nuovo Cimento*, **9**, 524 (1958).

[2] G. Cortini, G. Luzzatto, G. Tomasini, A. Manfredini: *Nuovo Cimento*, **9**, 706 (1958).

[3] C. Castagnoli, G. Cortini and A. Manfredini: *Nuovo Cimento*, **2**, 301 (1955).

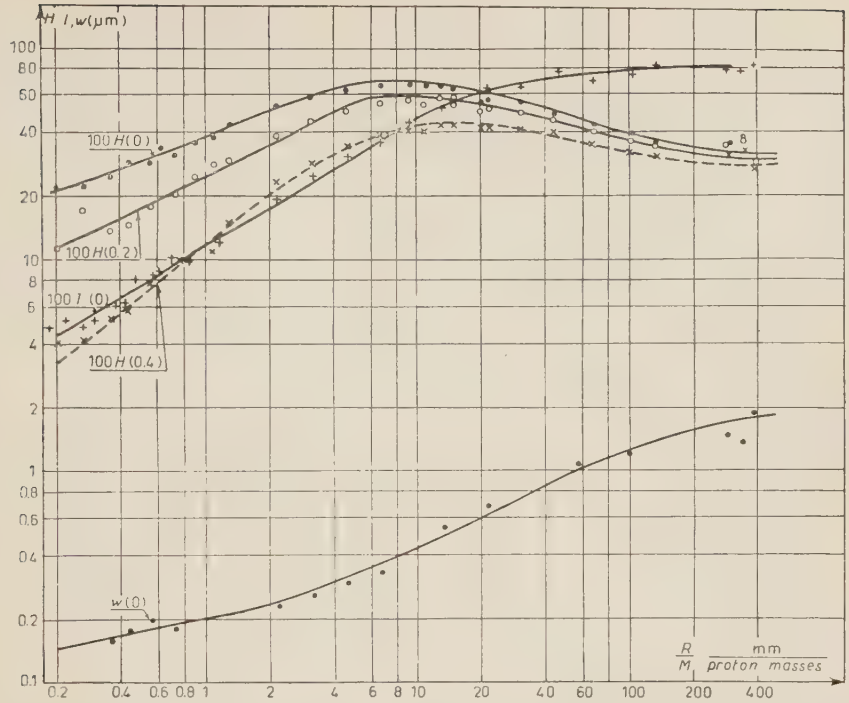

Fig. 1. – Measurement performed on the K-5 stack. The points until 20.2 mm/proton masses are obtained from proton tracks. From 21.5 mm/proton masses on, meson tracks were employed.

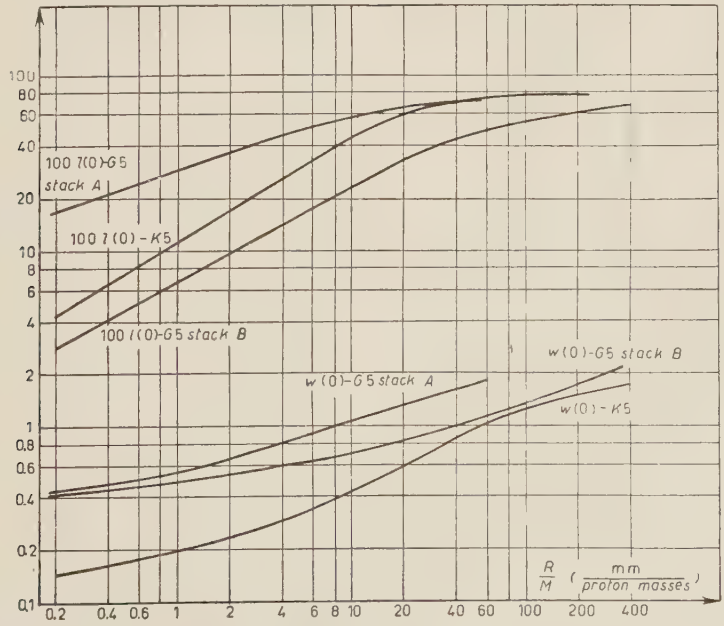

Fig. 2. – The $l(0)$ and $w(0)$ curves in 3 stacks: present K-5 stack; G-5-A stack-rather underdeveloped ([2]); G-5-B stack-strongly developed ([3]).

In a first analysis we neglect the variations of processing conditions: in our plates these give some remarkable trouble only near the edge of the plates; we neglect also the straggling of the range and the (supposed small) errors in the measurements of R.

Assume that the curve of the quantity f, plotted, as in Fig. 1 and 2 in a double logarithmic scale, be a straight line (within a certain range interval):

$$\lg f = a_f + b_f \lg (R/M) , \tag{6}$$

with a_f and b_f constants.

Then, we obtain

$$\frac{\delta M}{M} = \frac{1}{b} \frac{\delta f}{f} . \tag{7}$$

The standard fluctuation δf of f can be deduced, for any considered quantity from the Blatt « fluctuation parameters » ([2-4]). If one makes reference to a standard 100 μm measured cell, one gets the following formulae

$$\left\{ \begin{array}{l} \chi_B = \dfrac{\mu_B}{b_B \sqrt{100\ B}} ; \\ \chi_l = \dfrac{\mu_l}{b_l \sqrt{100\ l}} ; \\ \chi_w = \dfrac{\mu_w}{b_w \sqrt{100\ B}} , \end{array} \right. \tag{8}$$

where the suffixes indicate which is the measured quantity.

For the fluctuation parameters μ_B and μ_l we use the formulae given in a previous paper ([2]).

Besides, we take $\mu_w = 1$, as it would be for a pure exponential distribution. This approximation is possible for the rather high R/M values where the quantity w is a good measurement parameter (see Table I); indeed in this range region the distorsion of the gap length distribution curve with respect to a pure exponential is small, particularly in the K-5 plates (see ref. ([2])). In fact, experimental direct checks show that this procedure gives a slight overestimate of χ_w (as well of as χ_l, see ref. ([2])).

The results of the calculations are given in Table I. They are to be considered fully significant only in the regions where eq. (2) holds, that is where b is constant.

Note that the sudden strong increase in χ_l and χ_B with increasing R/M is due to the bending of the curves (decrease of the corresponding values of b).

However, we can draw from the table in a quantitative way, a number of useful conclusions about the best quantity to be measured in each case, and the discrimination power of each emulsion.

Particularly remarkable is the small value of χ_l for the K-5 plates between 5 and 20 mm of proton residual range.

In this region, under ideal conditions (absence of processing fluctuations) a mass measurement with a $\sim 15\%$ error could be made by measuring l on a 100 μm section, *i. e.* with a quarter-of-hour work.

In practice, for the proton measurements in this region we find experimentally a mass dispersion in good agreement with the calculated one.

However for the meson measurements, which have often been made near the edge of the plate, we find for l a dispersion larger than the calculated one by about a factor of 2. This corresponds to the larger fluctuation of points in the l-curve over 20 mm.

The measurements on the « A » stack, composed of several emulsions, give fluctuations somewhat higher than predicted by Table I. For 400 μm measurements of l the calculated values of $\delta M/M$ must be increased by $\sim(30 \div 40)\%$.

([4]) J. M. BLATT: *Australian Jour. Physics*, **8**, 248 (1955).

TABLE I. – *Values of the fluctuation parameters* χ_l, χ_B *and* χ_w *calculated from eqs.* (8) *and for the specificated measure quantities, in 3 different stacks.*

Stack	G-5 (A)			G-5 (B)			K-5		
f / R/M	l	B	w	l	B	w	l	B	w
.2	.41	.56		—	—	—	.59	.97	
.5	.37	.47		.52	.75		.34	.43	
1	.33	.40		.43	.64		.27	.27	
2	.29	.67	.44	.37	.53		.23	.23	.46
5	.24		.44	.30	.40		.17	.68	.28
10	.28		.45	.25	.61		.12	—	.26
20	.54		.47	.21			.13	—	.30
50			.51	.35		.55	>.40	—	.33
100	—	—	—	.65		.52	—	—	.42

Stacks G-5 A: underdeveloped. Stack G-5 B: very strongly developed. Stack K-5: slightly underdeveloped: plateau of 20 blob/100μm. The variable R/M is expressed in mm/proton masses. The white boxes correspond to very high X-values, the barred boxes to unmeasured values.

On the contrary, the w measurements give mass fluctuations which are in good agreement with the data of Table I. This is consistent with the view that the « O'Ceallaigh parameter » w is less dependent than others on the processing conditions ([3-5]).

It would be interesting to test if an appropriate underdevelopment could allow to shift the discussed favourable working conditions to smaller values of R/M.

Other interesting comparisons could be made with the L-4 Ilford plates having crystal radii still smaller than the K-5, as well as with other recently prepared fine-grained nuclear emulsions ([6-7])

* * *

Our sincere thanks are due to Prof. E. PANCINI for many useful discussions and to Prof. J. M. CASSELS for the exposure of the plates.

We are greatly indebted to Prof. C. FRANZINETTI and to Dr. M. FERRO-LUZZI for friendly help in arranging the exposure and processing of the plates.

([5]) C. O. CEALLAIGH: *Cern Secr. St. Meas.* n. 11 (1954).

([6]) P. DEMERS: *Can. Jour. Physics*, **8**, 538 (1954).

([7]) N. A. PERFILOV, N. R. NOVIKOVA and E. I. PROKOFIEVA: private communication, see also DENISENKO, N. S. IVANOVA, N. R. NOVIKOVA, N. A. PERFILOV, E. I. PROKOFIEVA and V. P. ŠAMOV: *Phys. Rev.*, **109**, 1779 (1958).

IL NUOVO CIMENTO Vol. IX, N. 6 16 Settembre 1958

Charged Meson - Nucleon Dispersion Relations with Fixed Nucleons.

A. Minguzzi

Istituto di Fisica dell'Università - Bologna
Istituto Nazionale di Fisica Nucleare - Sezione di Bologna

(ricevuto il 30 Luglio 1958)

The relativistic meson-nucleon dispersion relations in the forward direction have been proved by using in addition to the causality requirement, the hypothesis that the masses of charged and uncharged pions are the same. Let us consider the more general case of a pair of mesons with different masses both strongly coupled to the nucleon field. Following the procedure of Symanzik [1] it is straightforward to see that the conventional dispersion relations for a single mass meson continue to hold for the meson of lower mass; however for the meson of higher mass they still have to be proved. It follows that, due to the difference between the mass of the charged and uncharged mesons, the scattering dispersion relation are rigorously proved only for (π^0, N) scattering.

Following reference [1], one can easily trace the difficulties which are encountered in solving the problem when two masses are present. In the « unphysical range » we have besides the usual one particle states contribution from two particle states; the range of integration in the « unphysical range » is smaller the smaller the difference between the masses of the two mesons. Another difficulty arise from one particle states; in deriving dispersion relation an important role is played by the meson nucleon vertex operator. Its singularities in the complex energy plane can be identified and localized by the causality condition and the fact that only one meson mass is involved; a change in the spectrum amounts to additional unknown singularities; and these new singularities may or may not allow us to isolate the pole due to the « bound state »; we insist, however, that additional singularities of the meson nucleon vertex operator when two different masses are present does not necessarily imply a different contribution from the « bound state ». However in the limit of a fixed nucleon source this difficulty disappears; in this case normalizing the energy of the nucleon to zero, the energy spectrum is a point spectrum plus a continuum ranging from the lowest mass meson up to infinity. We will try to find the charged meson-nucleon dispersion relations is this limiting case.

---

[1] K. Symanzik: *Phys. Rev.*, **105**, 743 (1956).

Following reference [2] the scattering amplitude for $\pi^{\pm}$ on protons is

$$M(\omega, \boldsymbol{\Delta}) = \int \mathrm{d}^3x \exp[-i\boldsymbol{q}\cdot\boldsymbol{x}] \int_0^\infty \mathrm{d}x_0 \exp[i\omega x_0] \int \mathrm{d}^3y \exp[i\boldsymbol{\Delta}\cdot\boldsymbol{y}] \cdot$$

$$\cdot \frac{i}{2} \operatorname{Tr} \left\langle p \left| \left[J_{\pm}^{\dagger}\left(\boldsymbol{y} + \frac{\boldsymbol{x}}{2}, \frac{x_0}{2}\right), J_{\pm}\left(\boldsymbol{y} - \frac{\boldsymbol{x}}{2}, -\frac{x_0}{2}\right)\right] \right| p \right\rangle .$$

The combinations $M_{\pm} = \frac{1}{2}[M_{\pi^+} \pm M_{\pi^-}]$ in the limit of zero momentum transfer can be written

$$M_{\pm}(\omega) = \int \mathrm{d}^3x \exp[-i\boldsymbol{q}\cdot\boldsymbol{x}] F_{\pm}(\omega, x^2) , \tag{1}$$

with

$$F_{\pm}(\omega, x^2) = \frac{i}{4} \int_0^\infty \mathrm{d}x_0 \exp[i\omega x_0] \lim_{\boldsymbol{\Delta}\to 0} \operatorname{Tr} \left\{ \left\langle p \left| \left[J_{+}^{\dagger}\left(\boldsymbol{y} + \frac{\boldsymbol{x}}{2}, \frac{x_0}{2}\right), J_{+}\left(\boldsymbol{y} - \frac{\boldsymbol{x}}{2}, -\frac{x_0}{2}\right)\right] p \right\rangle \pm \right.\right.$$

$$\left.\left. \pm \left\langle p \left| \left[J_{-}^{\dagger}\left(\boldsymbol{y} + \frac{\boldsymbol{x}}{2}, \frac{x^0}{2}\right), J_{-}\left(\boldsymbol{y} - \frac{\boldsymbol{x}}{2}, -\frac{x_0}{2}\right)\right] p \right\rangle \right\} .$$

Due to the fact that $F_{\pm}(\omega, x^2)$ does not depend on the $\boldsymbol{x}$ direction, can we integrate (1) over the angles

$$M_{\pm}(\omega) = 4\pi \int_0^\infty \mathrm{d}x \frac{x \sin(\sqrt{\omega^2 - \mu^2 x})}{\sqrt{\omega^2 - \mu^2}} F_{\pm}(\omega, x^2) = \int_0^\infty \mathrm{d}x \, m_{\pm}(\omega, x^2) .$$

From the requirement of causality and the form of the spectrum, it follows

$$\text{(2a)} \quad \operatorname{Re} m_+(\omega_2, x^2) - \operatorname{Re} m_+(\omega_1, x^2) = \frac{2}{\pi}(\omega_2^2 - \omega_1^2) \int_{\mu_0}^{\mu} \frac{\omega' \operatorname{Im} m_+(\omega', x^2)}{(\omega_1^2 - \omega'^2)(\omega_2^2 - \omega'^2)} \mathrm{d}\omega' +$$

$$+ \frac{2}{\pi}(\omega_2^2 - \omega_1^2) \int_{\mu}^{\infty} \frac{\omega' \operatorname{Im} m_+(\omega', x^2)}{(\omega_1^2 - \omega'^2)(\omega_2^2 - \omega'^2)} \mathrm{d}\omega' ,$$

$$\text{(2b)} \quad \operatorname{Re} m_-(\omega_2, x^2) - \frac{\omega_2}{\omega_1} \operatorname{Re} m_-(\omega, x^2) = \frac{2\omega_2}{\pi}(\omega_2^2 - \omega_1^2) \int_{\mu_0}^{\mu} \frac{\operatorname{Im} m_-(\omega', x^2)}{(\omega_1^2 - \omega'^2)(\omega_2^2 - \omega'^2)} \mathrm{d}\omega' +$$

$$+ \frac{2\omega_2}{\pi}(\omega_2^2 - \omega_1^2) \int_{\mu}^{\infty} \frac{\operatorname{Im} m_-(\omega', x^2)}{(\omega_1^2 - \omega'^2)(\omega_2^2 - \omega'^2)} \mathrm{d}\omega' + \frac{\omega_2^2 - \omega_1^2}{\omega_1^2 \omega_2} B(x^2) ,$$

---

[2] R. OEHME: *Nuovo Cimento*, **4**, 1316 (1956).

where on the right hand side are written the contributions from the continuum and the point spectrum respectively. Even if the integration in (2*a*, *b*) over ω' from μ_0 to μ cannot be explicitly done, if we suppose that $m_{\pm}(\omega', x^2)$ is sufficiently regular when $\mu_0 < \omega' < \mu$ and $\omega_1 - \mu \gg \mu - \mu_0$, $(\omega_2 - \mu) \gg \mu - \mu_0$ we can at least show the dependence on ω_1, ω_2 by factoring the integral in the unphysical range of (2*a*) as

$$\int_{\mu_0}^{\mu} \frac{\omega' \operatorname{Im} m_+(\omega', x^2)}{(\omega_1^2 - \omega'^2)(\omega_2^2 - \omega'^2)} \, \mathrm{d}\omega' \sim \frac{\mu}{(\omega_1^2 - \mu^2)(\omega_2^2 - \mu^2)} \int_{\mu_0}^{\mu} \operatorname{Im} m_+(\omega'^2, x^2) \, \mathrm{d}\omega' .$$

The successive integration over x on the two sides of (2*a*), followed by the interchange of the x and ω' integration in the physical range of (2*a*), allows us to write

$$(3a) \quad D_+(\omega_2) - D_+(\omega_1) - \frac{2}{\pi}(\omega_2^2 - \omega_1^2) P \int_{\mu}^{\infty} \frac{\omega' A_+(\omega')}{(\omega_1^2 - \omega'^2)(\omega_2^2 - \omega'^2)} \, \mathrm{d}\omega' = \frac{\mu(\omega_2^2 - \omega_1^2)}{(\omega_1^2 - \mu^2)(\omega_2^2 - \mu^2)} \Lambda .$$

The boundedness of Λ is guaranteed by the boundedness of the left hand side of (3*a*). In the same way we can obtain:

$$(3b) \quad D_-(\omega_2) - \frac{\omega_2}{\omega_1} D_-(\omega_1) - \frac{2}{\pi} \omega_2 (\omega_2^2 - \omega_1^2) P \int_{\mu}^{\infty} \frac{A_-(\omega')}{(\omega_1^2 - \omega'^2)(\omega_2^2 - \omega'^2)} \, \mathrm{d}\omega' +$$

$$+ \frac{\omega_2(\omega_2^2 - \omega_1^2)}{(\omega_1^2 - \mu^2)(\omega_2^2 - \mu^2)} \Lambda = \frac{2(\omega_2^2 - \omega_1^2)}{\omega_1^2 \omega_2} \Gamma ,$$

where the boundedness of Γ is assured from the boundedness of the left hand side of (3*b*). If morever one supposes that $m_{\pi^-}(\omega', x^2)$ for $\mu_0 \leqslant \omega' < \mu$ can be developed in a uniformly convergent Taylor series around $\omega' = \mu$ and one takes only the first order contribution in $\mu - \mu_0$, one can express the Λ factor through the (π^-, p) charge exchange cross section and one obtains the results of Cini and Agodi [3].

The boundedness of Γ and Λ has been shown only *a posteriori*. A model which would have guaranteed their boundedness *a priori* would have been the one which assumes that the meson-nucleon interaction vanishes outside a finite region *i.e.* $J(r, t) = 0$ for $r > r_0$. The demonstrated boundedness of the Γ and Λ coefficients implies that the meson nucleon interaction falls off at least as rapidly as $\exp[-\mu r]$.

---

[3] A. Agodi and M. Cini: *Nuovo Cimento*, 5, 1256 (1957).

Note on the Electron Capture Decay of $^{121}Te^{m}$.

R. K. Gupta, S. Jha and B. K. Madan

Tata Institute of Fundamental Research - Bombay

(ricevuto il 18 Agosto 1958)

In a recent communication from this laboratory [1] it was suggested that the 1130 keV γ-ray occurring in the decay of ^{121}Te was emitted presumably from the 154 day isomeric state of ^{121}Te. A sample of metallic antimony irradiated in the deuteron beam in the cyclotron of the Birmingham University was obtained, from which pure tellurium was chemically separated. The γ-ray spectrum of the tellurium fraction was studied in a scintillation spectrometer for a period of over eight weeks. It was found that whereas the 575 keV γ-ray activity decayed with a half-life of about 17 days, the 1130 keV γ-ray intensity decayed with a much longer half-life. Taken together with the results reported earlier [1], it can be concluded that the 294 keV isomeric state of ^{121}Te known to be decaying with a half-life of 154 days by the emission of 214 keV γ-ray and a highly converted 80 keV γ-ray has another decay branch *i.e.* it decays in about 5% cases also by orbital electron capture to an excited state of ^{121}Sb giving rise to the 1130 keV γ-ray.

It may be pointed out that an examination of the systematics of the occurrence of the $d_{5/2}$ and the $g_{7/2}$ levels in the isotopes of Sb, Te, and I shows that a $g_{7/2}$ level in ^{121}Sb should lie a little above the ground state $d_{5/2}$ level. It seems to us that the 70 keV γ-ray found in the decay of ^{121}Te arises from a level at about 70 keV above the ground state. Unlike what has been assumed hitherto [1-3] the results of our studies are not inconsistent with the assumption that the 506 keV γ-ray originates in a transition from the 575 keV to the 70 keV level.

From a rough estimate of the decay energy involved in the 5% electron capture decay branching of $^{121}Te^{m}$, one finds that the log ft value for this decay is about 8. The coincidence study of these γ-rays and their internal conversion coefficients would be of great interest. Such studies are in progress.

---

[1] K. S. Bhatki, R. K. Gupta, S. Jha and B. K. Madan: *Nuovo Cimento*, **6**, 1461 (1957).

[2] M. Goldheber and R. D. Hill: *Rev. Mod. Phys.*, **24**, 179 (1952).

[3] M. Goldberg and S. Frankel: *Phys. Rev.*, **100**, 1350 (1955).

LIBRI RICEVUTI E RECENSIONI

J. L. Synge – *The Relativistic Gas* North Holland Pb. Co, Amsterdam, 1957, pag. xi-108.

In questo volumetto, che fa seguito al volume dell'autore sulla teoria speciale della relatività, si descrivono le proprietà che avrebbe un certo gas ideale in condizioni in cui è necessario usare la meccanica relativistica in luogo di quella di Newton. La trattazione presenta, dal punto di vista formale, una certa eleganza; ci sembra tuttavia che il titolo promettesse un po' di più che non della matematica ed è in questo senso che questa lunga applicazione dei metodi illustrati nel precedente volume sulla relatività speciale ci ha delusi. Bisogna dire che l'autore avverte nella prefazione che il libro « is written for the relativist who wants to know about the behaviour of a relativistic gas rather than for the expert in statistical mechanics ».

Grazie soprattutto all'eleganza formale di cui dicevamo sopra, la lettura è piuttosto gradevole. Particolarmente interessante è il capitolo dedicato alle onde d'urto in cui si vede come con una certa semplicità di mezzi matematici è possibile impostare e risolvere problemi complessi senza rinunciare alla generalità. Sono anche molto interessanti le considerazioni svolte al capitolo VI, riguardanti le leggi adiabatiche e l'equipartizione dell'energia: un po' infelice, forse, la distinzione « caldo » e « freddo », per un gas di particelle di massa di riposo m_0, fatta in base alla disuguaglianza $KT \gtrless m_0 C^2$.

Nel complesso, il libro, ripetiamo, delude. Però se qualcuno di buona volontà si proponesse di studiare un gas relativistico un po' più realistico del modello qui considerato, forse si troverebbe avvantaggiato leggendo questo libro ed il suo lavoro consisterebbe principalmente nell'estensione dei metodi quivi illustrati.

Carlo Bernardini

A. R. Meetham – *Basic Physics* Pergamon Press, London, 1957, pp. xi-144.

In questo libro di fisica elementare, destinato ad un pubblico senza specifiche nozioni in materia, viene esposto un numero considerevole di nozioni in uno stile estremamente conciso e sbrigativo. Questo metodo di esposizione rende assai problematica la comprensione degli argomenti esposti da parte di chi non abbia già una certa dimestichezza con la fisica. Risulta infatti difficile costruirsi attraverso la lettura del libro, un quadro sia pure semplice ma organico e logicamente collegato di questa scienza. Mancano inoltre chiare precisazioni sulle unità di misura. Queste limitazioni e talune inesattezze e oscurità nella esposizione, rendono discutibile l'utilità della lettura di questo libro.

A. Alberigi